Improper Integrals

$\int_a^b f(x)$ assumed $[a,b]$ is of finite length and f is continuous. What if $[a,\infty)$ or f is not continuous? These are improper integrals.

Basically, you find the integral as before, except you will have to find a limit at the end, if it exists.

Example 1: Find $\int_0^\infty e^{-x}dx$

Solution:

∞ is not a number so this is definitely an improper integral.

$$\int_0^a e^{-x}dx = \left.\frac{e^{-x}}{-1}\right|_0^a = \left[\frac{-1}{e^a}+\frac{1}{e^0}\right] = \frac{-1}{e^a}+1$$

If $\int_0^\infty e^{-x}dx$ is to exist then by the definition of a type I improper integral

$\lim\limits_{a\to\infty}\int_0^a e^{-x}dx$ must exist. ie. $\lim\limits_{a\to\infty}\left[\frac{-1}{e^a}+1\right]$ must exist.

$$\lim_{a\to\infty}\int_0^a e^{-x}dx = \lim_{a\to\infty}\left[\frac{-1}{e^a}+1\right] = -0+1=1$$

What does this really mean?

It states that as a increases in value the integral gets closer and closer to the number 1.

As an illustration

$$\int_0^1 e^{-x}dx = \frac{-1}{e}+1 \approx -0.36788+1 = 0.63212$$

$$\int_0^2 e^{-x}dx = \frac{-1}{e^2}+1 \approx -0.135335+1 = 0.864665$$

$$\int_0^{10} e^{-x}dx = \frac{-1}{e^{10}}+1 \approx -0.0000454+1 = 0.9999546$$

$$\int_0^{100} e^{-x}dx = \frac{-1}{e^{100}}+1 \approx -\left(3.72\times10^{-44}\right)+1 \approx 1$$

$\leftarrow$ As $a\to\infty$ the integral $\to 1$

A more difficult improper integral to recognize is a type II improper integral, with a discontinuity at an interior point.

Example 2: Find $\int_0^3 \frac{dx}{(x-1)^{\frac{2}{3}}}$, if it exists.

Solution:

There is no ∞ symbol to tip you off that the integral is improper. However, the function is discontinuous at $x = 1$.

$$\int_0^3 \frac{dx}{(x-1)^{\frac{2}{3}}} = \int_0^1 \frac{dx}{(x-1)^{\frac{2}{3}}} + \int_1^3 \frac{dx}{(x-1)^{\frac{2}{3}}} \text{ which by definition}$$

$$= \lim_{t\to 1^-} \int_0^t \frac{dx}{(x-1)^{\frac{2}{3}}} + \lim_{R\to 1^+} \int_R^3 \frac{dx}{(x-1)^{\frac{2}{3}}}$$

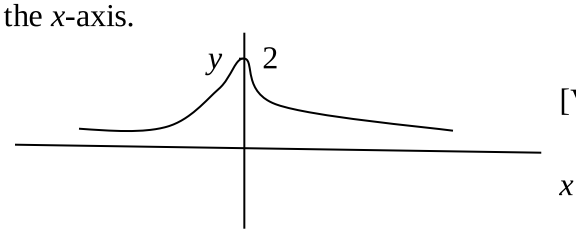
We will attempt to find these integrals separately. If any <u>one</u> of these limits does not exists we are done, the original integral <u>does not exist</u>.

$$\int_0^1 \frac{dx}{(x-1)^{\frac{2}{3}}} = \lim_{t\to 1^-} \int_0^t \frac{dx}{(x-1)^{\frac{2}{3}}} = \lim_{t\to 1^-} 3(x-1)^{\frac{1}{3}}\Big|_0^t = \lim_{t\to 1^-}\left[3(t-1)^{\frac{1}{3}} + 3\right] = 3$$

$$\int_1^3 \frac{dx}{(x-1)^{\frac{2}{3}}} = \lim_{R\to 1^+} \int_R^3 \frac{dx}{(x-1)^{\frac{2}{3}}} = \lim_{R\to 1^+} 3(x-1)^{\frac{1}{3}}\Big|_R^3 = \lim_{R\to 1^+}\left[3(3-1)^{\frac{1}{3}} - 3(R-1)^{\frac{1}{3}}\right] = 3\sqrt[3]{2}$$

$$\therefore \int_0^3 \frac{dx}{(x-1)^{\frac{2}{3}}} = 3 + 3\sqrt[3]{2}$$

Example 3: Find the area of the infinite region that lies under the curve $y = \frac{2}{1+x^2}$ and above the x-axis.



[Very rough sketch, should be symmetrical.]

$$\text{Area} = \int_{-\infty}^{\infty} \frac{2}{1+x^2}\, dx = 2\int_0^{\infty} \frac{2}{1+x^2}\, dx = 4\int_0^{\infty} \frac{1}{1+x^2}\, dx = 4\lim_{a\to\infty} \int_0^a \frac{1}{1+x^2}\, dx$$

$$= 4\lim_{a\to\infty} \tan^{-1} x\Big|_0^a = 4\left[\lim_{a\to\infty} \tan^{-1} a\right] - 4\tan^{-1} 0 = 4\left[\frac{\pi}{2}\right] - 0 = 2\pi$$

Example 4: Evaluate, if it exists, $\int_0^2 x^2 \ln x\,dx$

Solution:

$$\int x^2 \ln x\,dx = \frac{x^3}{3}\ln x - \int \frac{x^2}{3}\,dx \qquad\qquad let\ \ u = \ln x \qquad dv = x^2dx$$

$$= \frac{x^3}{3}\ln x - \frac{x^3}{9} + C \qquad\qquad du = \frac{1}{x}dx \qquad v = \frac{x^3}{3}$$

Therefore, $\int_0^2 x^2 \ln x\,dx = \lim_{a\to 0^+}\int_a^2 x^2 \ln x\,dx = \lim_{a\to 0^+}\left[\frac{x^3}{3}\ln x - \frac{x^3}{9}\right]_a^2 = \frac{8}{3}\ln 2 - \frac{8}{9} - \lim_{a\to 0^+}\left[\frac{a^3}{3}\ln a - \frac{a^3}{9}\right]$

$$= \frac{8}{3}\ln 2 - \frac{8}{9} - \lim_{a\to 0^+}\frac{a^3}{3}\ln a + 0$$

But

$$\lim_{a\to 0^+}\frac{a^3}{3}\ln a \qquad Form: (0\cdot -\infty)$$

$$= \lim_{a\to 0^+}\frac{\ln a}{3/a^3} \qquad Form: \left(\frac{-\infty}{\infty}\right)\ By\ L'Hopitals's\ rule$$

$$= \lim_{a\to 0^+}\frac{1/a}{-9/a^2} = \lim_{a\to 0^+}\left(-\frac{a}{9}\right) = 0$$

Hence $\int_0^2 x^2 \ln x\,dx = \frac{8}{3}\ln 2 - \frac{8}{9} - 0 + 0 = \frac{8}{3}\ln 2 - \frac{8}{9}$

Example 5: Evaluate $\int_0^\infty \frac{x}{1+x^2}\,dx$

Solution:

$$\int_0^\infty \frac{x}{1+x^2}\,dx = \lim_{a\to\infty}\int_0^a \frac{x}{1+x^2}\,dx = \frac{1}{2}\lim_{a\to\infty}\int_0^a \frac{2x}{1+x^2}\,dx = \frac{1}{2}\lim_{a\to\infty}\left[\ln\left(1+x^2\right)\right]_0^a$$

$$= \frac{1}{2}\lim_{a\to\infty}\left[\ln\left(1+x^2\right)\right]_0^a = \frac{1}{2}\lim_{a\to\infty}\ln\left(1+a^2\right) - \frac{1}{2}\ln\left(1+0^2\right) = \frac{1}{2}\infty - \frac{1}{2}(0) = does\ not\ exist$$

Therefore, $\int_0^\infty \frac{x}{1+x^2}\,dx$ diverges.

Type I and type II are the two types of improper integrals we will study. However, we must now discuss the Comparison Test. In this section you are not asked to find the integral you are only to determine whether the integral exists or does not exist. The method is very similar to the limit Squeeze Theorem you encountered in calculus 1500.

We will restrict ourselves to only integrals which have a positive value, if they exist. The strategy is as follows:

We are asked to determine whether $\int_a^b f(x)\,dx$ does or does not exist.

If we can find, or know, of another integral $\int_a^b g(x)\,dx$ such that

(a) $0 \le \int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$ and $\int_a^b g(x)\,dx$ converges then $\int_a^b f(x)\,dx$ converges. Basically,

(b) $0 \le \int_a^b g(x)\,dx \le \int_a^b f(x)\,dx$ and $\int_a^b g(x)\,dx$ diverges then $\int_a^b f(x)\,dx$ diverges.

if our integral is between 0 and another number then our integral is a number. Whereas, if our integral is greater than ∞ than its is also gigantic.

It is important to have a reservoir of these known integrals $\int_a^b g(x)\,dx$ which we can use to trap our integral $\int_a^b f(x)\,dx$, or show that our integral $\int_a^b f(x)\,dx$ is greater than this divergent $\int_a^b g(x)\,dx$!!!!!

The two most common integrals we use for comparison are the p-integrals and the exponential function integral.

p-integral test

The integrals, $\int_1^\infty \frac{1}{x}\,dx$ and $\int_0^1 \frac{1}{x}\,dx$ both diverge. They get larger and larger at both ends.

The p-integrals are: $\int_1^\infty \frac{1}{x^p}\,dx$ and $\int_0^1 \frac{1}{x^p}\,dx$. The p-test states:

(a) if $0 \le \int_1^\infty \frac{1}{x^p}\,dx < \int_1^\infty \frac{1}{x}\,dx$ then $\int_1^\infty \frac{1}{x^p}\,dx$ is convergent.

(b) if $0 \le \int_1^\infty \frac{1}{x}\,dx < \int_1^\infty \frac{1}{x^p}\,dx$ then $\int_1^\infty \frac{1}{x^p}\,dx$ is diverges.

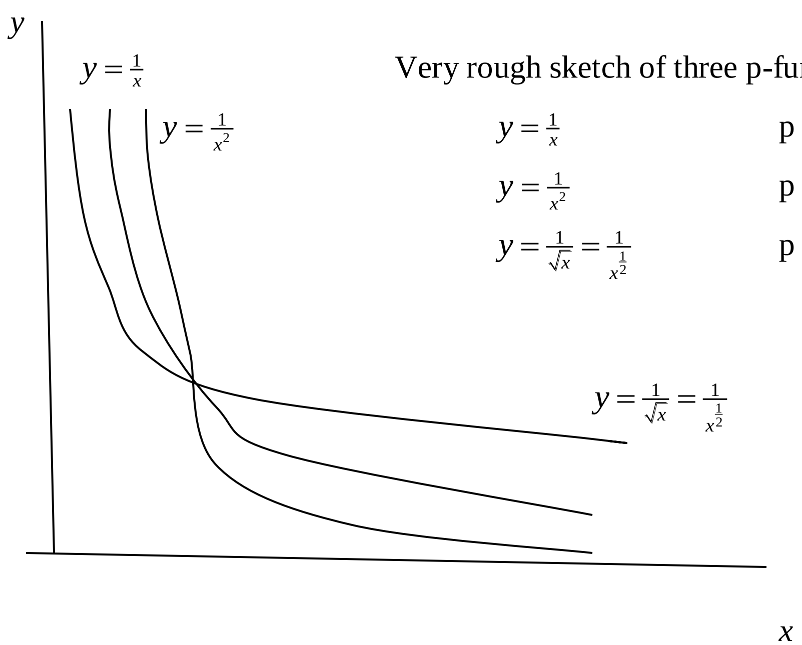
(c) if $0 \le \int_0^1 \frac{1}{x^p}\,dx < \int_0^1 \frac{1}{x}\,dx$ then $\int_0^1 \frac{1}{x^p}\,dx$ is convergent.

(d) if $0 \le \int_0^1 \frac{1}{x}\,dx < \int_0^1 \frac{1}{x^p}\,dx$ then $\int_0^1 \frac{1}{x^p}\,dx$ is divergent.

A more concise way of stating the p-integral test is:

(a) $\int_1^\infty \frac{1}{x^p}\,dx$ converges provided p>1, diverges otherwise.

(a) $\int_0^1 \frac{1}{x^p}\,dx$ converges provided p<1, diverges otherwise.



Very rough sketch of three p-functions $y = \frac{1}{x^p}$

$y = \frac{1}{x}$	p = 1
$y = \frac{1}{x^2}$	p = 2 >1
$y = \frac{1}{\sqrt{x}} = \frac{1}{x^{\frac{1}{2}}}$	p = $\frac{1}{2}$<1

[Note: There is nothing sacred about the number 1. The key values are 0 and ∞.

Hence, *a* can be any positive number.]

The <u>second</u> very common integral used for comparison is: $\int_0^\infty e^{-ax}\,dx = \int_0^\infty \frac{1}{e^{ax}}\,dx \;\; for\; a > 0$

This integral always converges.

Example 1: Use the comparison test to determine whether $\int_1^\infty \frac{dx}{\sqrt{x^3+1}}\,dx$

converges or diverges.

Solution:

For $x \geq 1$, $\frac{1}{\sqrt{x^3+1}} < \frac{1}{\sqrt{x^3}} = \frac{1}{x^{\frac{3}{2}}} = \frac{1}{x^p}$ where $p = \frac{3}{2} > 1$

$\therefore 0 \leq \int_1^\infty \frac{1}{\sqrt{x^3+1}}\,dx < \int_1^\infty \frac{1}{x^{\frac{3}{2}}}\,dx$

By p-integral test $\int_1^\infty \frac{1}{x^{\frac{3}{2}}}\,dx$ converges, since $p = \frac{3}{2} > 1$

$\therefore$ By comparison test $\int_1^\infty \frac{1}{\sqrt{x^3+1}}\,dx$ converges.

[Note: We have not found the value of the integral. We have just determined that the value exists.]

Example 2: Use the comparison test to determine whether $\int_0^\infty \frac{dx}{e^x + x}$

converges or diverges.

Solution:

[Note: At $x = 0$ there is no problem since $e^0 + 0 = 1 + 0 = 1$.]

For $x \geq 0$, $\quad 0 \leq \dfrac{1}{e^x + x} \leq \dfrac{1}{e^x}$

But $\int_0^\infty \dfrac{1}{e^x} dx$ is convergent

$\therefore$ By comparison test $\int_0^\infty \dfrac{dx}{e^x + x}$ is also convergent.

Example 3: Use the comparison test to determine whether $\int_0^\infty \dfrac{dx}{\sqrt{x + x^3}}$ converges or diverges.

Solution:

There is a problem at both ends since the denominator is 0 for $x = 0$.

$$\int_0^\infty \frac{dx}{\sqrt{x+x^3}} dx = \int_0^1 \frac{dx}{\sqrt{x+x^3}} + \int_1^\infty \frac{dx}{\sqrt{x+x^3}}$$

For $x \geq 1$ $\quad 0 < \dfrac{1}{\sqrt{x+x^3}} < \dfrac{1}{\sqrt{x^3}} = \dfrac{1}{x^{\frac{3}{2}}} = \dfrac{1}{x^p}$ with $p = \dfrac{3}{2} > 1$

By p-integral test $\int_1^\infty \dfrac{dx}{x^{\frac{3}{2}}}$ converges (since $p = \dfrac{3}{2} > 1$)

$\therefore$ By comparison test $\int_1^\infty \dfrac{dx}{\sqrt{x+x^3}} dx$ also converges.

For $0 < x \leq 1$ $\quad 0 < \dfrac{1}{\sqrt{x+x^3}} < \dfrac{1}{\sqrt{x}} = \dfrac{1}{x^{\frac{1}{2}}} = \dfrac{1}{x^p}$ with $p = \dfrac{1}{2} < 1$

By p-integral test $\int_0^1 \dfrac{dx}{x^{\frac{1}{2}}}$ converges (since $p = \dfrac{1}{2} < 1$)

$\therefore$ By comparison test $\int_0^1 \dfrac{dx}{\sqrt{x+x^3}} dx$ also converges.

Hence, from the above two facts $\int_0^\infty \dfrac{dx}{\sqrt{x+x^3}}$ converges.

In the previous example, you should notice, that it is very important which integral you choose for comparison. Concluding that $0 < \frac{1}{\sqrt{x^3+x}} < \frac{1}{\sqrt{x^3}}$ is of absolutely no use if you are in the interval (0, 1], since being smaller than a divergent integral tells you nothing. Similarly, concluding that $0 < \frac{1}{\sqrt{x^3+x}} < \frac{1}{\sqrt{x}} = \frac{1}{x^{\frac{1}{2}}}$ is also of no use in the interval $[1, \infty)$. Being smaller than a divergent integral is also of absolutely no value.

<u>Homework:</u>

Use the comparison test to determine whether the integral is convergent or divergent.

(1) $\int_1^\infty \frac{x+1}{x^2}\,dx$ (2) $\int_1^\infty \frac{\sqrt{1+\sqrt{x}}}{\sqrt{x}}\,dx$ (3) $\int_0^1 \frac{e^{-x}}{\sqrt{x}}\,dx$

(4) $\int_1^\infty \frac{x^3}{x^5+2}\,dx$ (5) $\int_0^\infty \frac{e^x}{2x+1}\,dx$ (6) $\int_1^\infty e^{-x^2}\,dx$

(7) $\int_1^\infty \frac{1}{\sqrt{x}+1}\,dx$ (8) $\int_0^1 \frac{|\sin x|+1}{\sqrt{x}}\,dx$

Solutions: Be sure to try answering the question fully yourself before checking the solutions below.

(1) For $x \geq 1$, $\frac{x+1}{x^2} > \frac{x}{x^2} = \frac{1}{x} = \frac{1}{x^p}$ where $p = 1$

By p-integral test $\int_1^\infty \frac{1}{x}\,dx$ diverges.

and $0 < \int_1^\infty \frac{1}{x}\,dx \leq \int_1^\infty \frac{x+1}{x^2}\,dx$,

so by comparison test $\int_1^\infty \frac{x+1}{x^2}\,dx$ diverges.

(2) For $x \geq 1$, $\dfrac{\sqrt{1+\sqrt{x}}}{\sqrt{x}} > \dfrac{1}{\sqrt{x}} = \dfrac{1}{x^{\frac{1}{2}}} = \dfrac{1}{x^p}$ where $p = \dfrac{1}{2} < 1$

By p-integral test $\displaystyle\int_1^\infty \frac{1}{\sqrt{x}}\,dx$ diverges.

and $0 < \displaystyle\int_1^\infty \frac{1}{\sqrt{x}}\,dx \leq \int_1^\infty \frac{\sqrt{1+\sqrt{x}}}{\sqrt{x}}\,dx$,

so by comparison test $\displaystyle\int_1^\infty \frac{\sqrt{1+\sqrt{x}}}{\sqrt{x}}\,dx$ diverges.

(3) For $0<x \leq 1$, $\dfrac{e^{-x}}{\sqrt{x}} = \dfrac{1}{e^x\sqrt{x}} < \dfrac{1}{\sqrt{x}} = \dfrac{1}{x^{\frac{1}{2}}} = \dfrac{1}{x^p}$ where $p = \dfrac{1}{2}$

By p-integral test $\displaystyle\int_0^1 \frac{1}{\sqrt{x}}\,dx$ converges (since $p = \dfrac{1}{2} < 1$)

and $0 < \displaystyle\int_0^1 \frac{e^{-x}}{\sqrt{x}}\,dx \leq \int_0^1 \frac{1}{\sqrt{x}}\,dx$,

so by comparison test $\displaystyle\int_0^1 \frac{e^{-x}}{\sqrt{x}}\,dx$ converges.

(4) For $x \geq 1$, $\dfrac{x^3}{x^5+2} < \dfrac{x^3}{x^5} = \dfrac{1}{x^2} = \dfrac{1}{x^p}$ where $p = 2$

By p-integral test $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$ converges. (since $p = 2$.)

and $0 < \displaystyle\int_1^\infty \frac{x^3}{x^5+2}\,dx \leq \int_1^\infty \frac{1}{x^2}\,dx$,

so by comparison test $\displaystyle\int_1^\infty \frac{x+1}{x^2}\,dx$ converges.

(5) $\displaystyle\lim_{x\to\infty} \frac{e^x}{2x+1}$ $\left[\dfrac{\infty}{\infty}\right]$

$\underset{=}{[H]} \displaystyle\lim_{x\to\infty} \frac{e^x}{2} = \infty \neq 0$ $\quad \therefore \displaystyle\int_0^\infty \frac{e^x}{2x+1}\,dx$ diverges.

N.B. <u>An important fact</u>: If $\displaystyle\int_a^\infty f(x)\,dx$ converges then $\displaystyle\lim_{x\to\infty} f(x)\,dx = 0$

(6) For $x \le 1, e^{-x^2} = \dfrac{1}{e^{x^2}} \le \dfrac{1}{e^x}$

But $\int_1^\infty \dfrac{1}{e^x}\,dx$ converges

and $0 < \int_1^\infty e^{-x^2}\,dx \le \int_1^\infty \dfrac{1}{e^x}\,dx,$

so by comparison test $\int_1^\infty e^{-x^2}dx$ also converges.

(7) For $x \ge 1, \sqrt{x}+1 \le \sqrt{x}+\sqrt{x} = 2\sqrt{x}$

But $\int_1^\infty \dfrac{1}{2\sqrt{x}}\,dx = \dfrac{1}{2}\int_1^\infty \dfrac{1}{\sqrt{x}}\,dx$ diverges by p-integral test (p = $\dfrac{1}{2}$).

and $0 < \int_1^\infty \dfrac{1}{\sqrt{x}+1}\,dx \ge \dfrac{1}{2}\int_1^\infty \dfrac{1}{\sqrt{x}}\,dx,$

so by comparison test $\int_1^\infty \dfrac{1}{\sqrt{x}+1}$ also diverges.

[Note: This is a very useful technique when a constant is added or subtracted from a well know function.]

(8) For $x \ge 1, |\sin x| + 1 \le 1 + 1 = 2$

But $\int_0^1 \dfrac{2}{\sqrt{x}}\,dx = 2\int_0^1 \dfrac{1}{\sqrt{x}}\,dx$ converges by p-integral test (p = $\dfrac{1}{2}$).

and $0 < \int_0^1 \dfrac{|\sin x|+1}{\sqrt{x}}\,dx \le 2\int_0^1 \dfrac{1}{\sqrt{x}}\,dx,$

so by comparison test $\int_0^1 \dfrac{|\sin x|+1}{\sqrt{x}}\,dx$ also converges.

[Note: $\sin x$ and/or $\cos x$ often play the role of a constant between -1 and 0.]