Representation of functions as power series

By the sum of a geometric series, we have

$$\frac{1}{1-x} = 1 + x + x^2 + \dots = \sum_{n=0}^{\infty} x^n \qquad |x| < 1$$

then

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

$$= 1 - x^2 + x^4 - x^6 + \dots, \qquad |x^2| < 1$$

$\therefore$ Interval of convergence is $(-1, 1)$

$$\frac{1}{3+x} = \frac{1}{3}\left(\frac{1}{1+\frac{x}{3}}\right) = \frac{1}{3}\left(\frac{1}{1-(-x/3)}\right)$$

$$= \frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{-x}{3}\right)^n = \frac{1}{3}\sum_{n=0}^{\infty}\frac{(-1)^n x^n}{3^n} = \sum_{n=0}^{\infty}\frac{(-1)^n x^n}{3^{n+1}} = I$$

$$\lim_{n\to\infty}\left|\frac{(-1)^{n+1} x^{n+1}}{3^{n+2}} \cdot \frac{3^{n+1}}{(-1)^n x^n}\right| = \lim_{n\to\infty}\left|\frac{x}{3}\right|$$

$\therefore$ I is convergent if $|x| < 3$ and the interval of convergence is $(-3, 3)$

$$\frac{x^3}{x+3} = x^3 \cdot \frac{1}{x+3} = x^3 \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{3^{n+1}}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+3}}{3^{n+1}}$$

If a function $f(x) = \sum c_n (x-a)^n$ is defined by a power series with a radius of convergence $R > 0$, then f is differentiable (and continuous) on the interval $(a-R, a+R)$ and

a) $f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \ldots$

$$= \sum_{n=1}^{\infty} n c_n (x-a)^{n-1}$$

b) $\int f(x)\,dx = C + c_0(x-a) + \frac{c_1 (x-a)^2}{2} + \ldots$

$$= C + \sum_{n=0}^{\infty} \frac{c_n}{n+1} (x-a)^{n+1}$$