an <u>ordinary differential equation</u> (ODE) is an equation that involves an unknown function of one variable, and its derivatives (first, second, third, ...). The <u>order</u> of an ODE is the order of the highest derivative in the ODE.

Note: $\frac{dy}{dx} = y'$ $\frac{d^2y}{dx^2} = y''$

Examples

$y' = \cos(x+y)$ first order

$y'' = y^2x + xy + y'$ second order

$\frac{d^3y}{dx^3} + 3\frac{dy}{dx} = 7xy$ third order

Goal: Given an ODE, find y that satisfies the ODE.

Example: $y'' - y' + y = e^x(x+1)$

Note $y = xe^x$ is a solution because
$y' = e^x + xe^x$
$y'' = e^x + e^x + xe^x$ $\therefore y'' - y' + y = e^x + xe^x = e^x(x+1)$

Verifying a solution is easy. Finding a solution is hard.

Growth/Decay equation/model.

Rate of change of $y = \frac{dy}{dt}$, i.e. proportional to y,

$\frac{dy}{dt} = ky$ where $k \in \mathbb{R}$ or k is a constant.

If $k > 0$, then $\frac{dy}{dt} > 0$ always and you have unrestricted growth

If $k < 0$, then $\frac{dy}{dt} < 0$ always and you have radioactive decay

If $k = 0$, then $\frac{dy}{dt} = 0$ and the population stays constant.

The general solution to the growth/decay equation is $y = Ce^{kt}$, C constant.

Check: $y' = k \cdot C \cdot e^{kt} = k \cdot y$ ✓

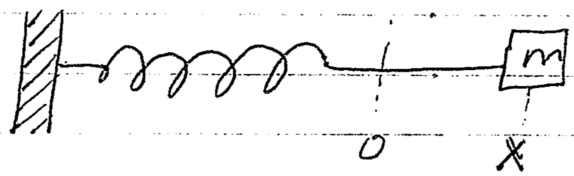
Rather than having unrestricted population growth, a more realistic model is the logistic model.

$y' = ay(1 - \frac{y}{b})$ or $\frac{dP}{dt} = kP(1 - \frac{P}{M})$

where $P(t)$ is the population at time t,
M is the carrying capacity

Note if $P_0 < M$ then $\frac{dP}{dt} > 0$
$P > M$ then $\frac{dP}{dt} < 0$

Physics Example



Hooke's law: restoring force $= -kx$ where k is the spring constant.

$F = ma$ (Newton's Law)

$$-kx = m \cdot \frac{d^2x}{dt^2} \Rightarrow x'' = \left(\frac{-k}{m}\right)x$$

Simple Harmonic Oscillator

Let $j = \left(\frac{k}{m}\right)^{1/2}$ then for any constant C.

$X = C\sin(jt)$ is a solution.

$x' = jC\,\cancel{\sin}\,\cos(jt)$ $\qquad x'' = j^2\,C(-(\sin(jt)))$

$$= -\frac{k}{m}\,C\sin(jt)$$

$$= -k/m\,X.$$

Similarly $X = C\cos(jt)$ is a solution.

$\therefore$ The general solution is $\quad C_1\sin(jt) + C_2\cos(jt)$

Often we want to solve an ODE subject to an initial condition.

Ex: $\frac{dy}{dx} = 2y \quad y(0) = 4$

The general solution is $y = Ce^{2x}$

we want $y(0) = 4 \quad 4 = y(0) = Ce^{2(0)} = C \quad \therefore C = 4$

$\therefore$ general solution subject to the Initial Value Condition is $4e^{2x}$.