Find the Maclaurin series of $f(x)=e^x$ and it's radius of convergence.

$$f^{(n)}(x) = e^x \qquad f^{(n)}(0) = e^0 = 1$$

$$e^x = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)\, x^n}{n!} = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Does this series truely converge to e^x?

The n^{th} degree Taylor Polynomial of f at a is the n^{th} partial sum of the Taylor series.

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{n!}(x-a)^k = f(a) + \frac{f'(a)}{1}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

The n^{th} remainder of the Taylor series is the difference $R_n(x) = f(x) - T_n(x)$.

The Taylor series converges to $f(x)$ at $x=x_0$ iff $\lim_{n\to\infty} R_n(x_0) = 0$.

Theorem (Taylor's Inequality)

If $|f^{(n+1)}(x)| \le M$ for $|x-a| \le d$, then

$$|R_n(x)| \le \frac{M}{(n+1)!}|x-a|^{n+1} \quad \text{for } |x-a| \le d.$$

Prove that e^x is equal to its Maclaurin Series.

$$|f^{(n+1)}(x)| = e^x \le e^d \quad \text{for } |x| \le d$$

$$\therefore \quad 0 \le |R_n(x)| \le \frac{e^d}{(n+1)!}|x|^{n+1} \quad \text{for } |x| \le d$$

$\therefore \lim_{n\to\infty} |R_n(x)| = 0$ by squeeze theorem since

$$\lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} = 0.$$

Ex Find the Maclaurin Series for $\sin(x)$.

$$
\begin{aligned}
f(x) &= \sin(x) & f(0) &= 0 \\
f'(x) &= \cos(x) & f'(0) &= 1 \\
f''(x) &= -\sin(x) & f''(0) &= 0 \\
f'''(x) &= -\cos(x) & f'''(0) &= -1 \\
f''''(x) &= \sin(x) & f''''(0) &= 0
\end{aligned}
$$

$$f(x) = f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \frac{f'''(0)}{3!} x^3 + \dots$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \qquad R = \infty.$$

$$|R_n(x)| \le \frac{|x|^{n+1}}{(n+1)!} \qquad \therefore R_n(x) \to 0 \text{ as } n \to \infty.$$

Find the Maclaurin Series for $\cos(x)$,

$$\cos(x) = \frac{d}{dx}(\sin(x)) = \frac{d}{dx}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \right)$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

let $f(x) = x^{1/2}$ $x > 0$
Calculate the 2^{nd} order polynomial at $x = 4$,

$f(4) = 2$
$f'(x) = \frac{1}{2} x^{-1/2}$ $f'(4) = 1/4$
$f''(x) = \frac{-1}{4} x^{-3/2}$ $f''(4) = -1/32$

$$T_2(x) = 2 + \frac{1}{4}(x-4) - \frac{1}{32 \cdot 2}(x-4)^2$$

How accurate is this approximation on $3 \leq x \leq 5$
$3 \leq x \leq 5 \Rightarrow |x-4| \leq 1$

$$|f'''(x)| = \frac{3}{8} x^{-5/2} = M$$

$$|R_2(x)| \leq \frac{M}{(2+1)!} |x-4|^3 \leq \frac{M}{3!} = \frac{M}{6} \approx .004,$$