Binomial Series.

Find the Maclaurin series for $f(x)=(1+x)^k$.

$f(x)=(1+x)^k$ $\qquad f(0)=1$

$f'(x)= k(1+x)^{k-1}$ $\qquad f'(0)=k$

$f''(x)= k(k-1)(1+x)^{k-2}$ $\qquad f''(0)=k(k-1)$

$f'''(x)= k(k-1)(k-2)(1+x)^{k-3}$ $\qquad f'''(0)=k(k-1)(k-2)$

$\vdots$

$f^{(n)}(x)= k(k-1)(k-2)\ldots(k-n+1)(1+x)^{k-n}$

$f^{(n)}(0)=k(k-1)\ldots(k-n+1)$

$$(1+x)^k=\sum_{n=0}^{\infty}\binom{k}{n}x^n = 1+kx+\frac{k(k-1)}{2!}x^2+\ldots$$

$$\binom{k}{n}=\frac{k!}{n!(k-n)!}=\frac{k(k-1)\ldots(k-n+1)}{n!}$$

By ratio test we get $R=1$.

Find the Maclaurin Series for

$$\frac{1}{\sqrt{r^2-x^2}}.$$

$$\frac{1}{\sqrt{r^2-x^2}} = \frac{1}{r}\frac{1}{\sqrt{1-x^2/r^2}} = \frac{1}{r}\left(1+\left(\frac{-x^2}{r^2}\right)\right)^{-1/2}$$

$$f(x) = \frac{1}{r}\sum_{n=0}^{\infty}\binom{-1/2}{n}\left(\frac{-x^2}{r^2}\right)^n$$

$$= \frac{1}{r}\sum_{n=0}^{\infty}\frac{\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)\cdots\left(-\frac{2n-1}{2}\right)(-1)^n x^{2n}}{n!\, r^{2n}}$$

$$= \sum_{n=0}^{\infty}\frac{1\cdot3\cdot5\cdot\ldots\cdot(2n-1)}{2^n(n!)}\frac{x^{2n}}{r^{2n+1}}$$

This is convergent if $\left|\frac{-x^2}{r^2}\right| < 1$ $\therefore R = r.$

List of important Maclaurin Series with their R

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1+x+x^2+\dots \qquad R=1$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\dots \qquad R=\infty$$

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x-\frac{x^3}{3!}+\frac{x^5}{5!}-\dots \qquad R=\infty$$

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\dots \qquad R=\infty$$

$$\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\dots \qquad R=1$$

$$\ln(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1} = x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\dots \qquad R=1$$

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1+kx+\frac{k(k-1)}{2!}x^2+\frac{k(k-1)(k-2)}{3!}x^3+\dots$$

$$R=1$$

Application

Find the sum of $s = \frac{1}{1\cdot 2} - \frac{1}{2\cdot 2^2} + \frac{1}{3\cdot 2^3} - \cdots$

$$s = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)2^{n+1}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}\cdot\left(\frac{1}{2}\right)^{n+1} = \ln\left(1+\frac{1}{2}\right)$$

Evaluate $\lim_{x\to 0} \frac{e^x - 1 - x}{x^2}$

$$= \lim_{x\to 0} \frac{(1 + x + x^2/2! + x^3/3! + \cdots) - 1 - x}{x^2}$$

$$= \lim_{x\to 0} \frac{x^2/2! + x^3/3! + \cdots}{x^2}$$

$$= \lim_{x\to 0} \frac{1}{2!} + \frac{x}{3!} + \frac{x^2}{4!} + \cdots$$

$$= \frac{1}{2}$$