Arc Length

You are expected to know the following formulae:

Cartesian

$$L=\int_a^b\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$$

$$L=\int_c^d\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

Parametric

$$L=\int_{t_1}^{t_2}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$$

Polar

$$L=\int_{\theta_1}^{\theta_2}\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta$$

$$L=\int_{\theta_2}^{\theta_1}\sqrt{\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2}\,d\theta$$

Some good questions:

Find the circumference of a circle with radius *a*.

Solution:

(a) Cartesian

"sketch"

$x^2+y^2=a^2 \qquad \therefore y=\pm\sqrt{a^2-x^2}$

Using the first quadrant we have $y=\sqrt{a^2-x^2}$

$$\therefore \frac{dy}{dx}=\frac{1}{2}\left(a^2-x^2\right)^{-\frac{1}{2}}(-2x)=\frac{-x}{\sqrt{a^2-x^2}}$$

$$Circumference=4\int_0^a\sqrt{1+\left(\frac{-x}{\sqrt{a^2-x^2}}\right)^2}\,dx=4\int_0^a\sqrt{1+\frac{x^2}{a^2-x^2}}\,dx$$

$$=4\int_0^a\sqrt{\frac{a^2-x^2+x^2}{a^2-x^2}}\,dx=4a\int_0^a\frac{1}{\sqrt{a^2-x^2}}\,dx$$

But this is an improper integral since $a^2-x^2=0$ when $x=a$.

$$\therefore Circumference=4a\lim_{R\to a^-}\int_0^a\frac{1}{\sqrt{a^2-x^2}}\,dx=4a\lim_{R\to a^-}\left[\sin^{-1}\frac{x}{a}\right]_0^R$$

$$=4a\left(\frac{\pi}{2}-0\right)=2\pi a \qquad \leftarrow "C=2\pi r"$$

(b) Parametric

"sketch"

parametric equations of a circle of radius $=a$ are:

$$x = a\cos t \qquad y = a\sin t$$

$$\frac{dx}{dt} = -a\sin t \quad \frac{dy}{dt} = a\cos t$$

$$L = 4\int_0^{\frac{\pi}{2}} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = 4\int_0^{\frac{\pi}{2}} \sqrt{\left(-a\sin t\right)^2 + \left(a\cos t\right)^2}\,dt$$

$$= 4\int_0^{\frac{\pi}{2}} \sqrt{a^2\left(\sin^2 t + \cos^2 t\right)}dt = 4\int_0^{\frac{\pi}{2}} a\,dt = 4\,at\Big|_0^{\frac{\pi}{2}} = 4a\left(\frac{\pi}{2}\right) = 2\pi a$$

(c) Polar

"sketch"

Polar equation of a circle with radius $= a$ is $r = a$

$$\therefore \frac{dr}{d\theta} = 0$$

Hence, $\quad Circumference = 4\int_0^{\frac{\pi}{2}} \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2}\,d\theta = 4\int_0^{\frac{\pi}{2}} \sqrt{\left(0\right)^2 + a^2}\,d\theta$

$$= 4\int_0^{\frac{\pi}{2}} a\,d\theta = 4a\,\theta\Big|_0^{\frac{\pi}{2}} = 4a\left(\frac{\pi}{2} - 0\right) = 2\pi a$$

Which one of the three systems seems to be the most efficient?

Some examples:

Example: (pg 552 #18) Set up the integral for the length of $y = 2^x$ on $0 \le x \le 3$.

Solution:

$$L = \int_0^3 \sqrt{1 + \left(2^x \ln 2\right)^2}\,dx$$

Example: Set up the integral that represents the length of the parametric curve

$$x = 1 + e^t,\ \ y = t^2 \qquad -3 \le t \le 3$$

Solution:

$$\frac{dx}{dt} = e^t \qquad \frac{dy}{dt} = 2t$$

$$\therefore L = \int_{-3}^{3} \sqrt{\left(e^t\right)^2 + \left(2t\right)^2}\, dt$$