Area of Polar Curves

Area between two polar curves:
It is very important that you sketch the curves on one polar system.

Example: Set up the integral which represents the area inside the curve $r = 2 - 2\sin\theta$ and outside $r = 3$.

Solution:
"graph"

Intersection:

$$3 = 2 - 2\sin\theta$$

$$2\sin\theta = -1$$

$$\sin\theta = -\frac{1}{2} \qquad \text{"}sketch\text{"}$$

$$\theta = \frac{7\pi}{6}, \frac{11\pi}{6}$$

$$area \quad = 2\int_{\frac{7\pi}{6}}^{\frac{3\pi}{2}} \frac{1}{2}(2 - 2\sin\theta)^2 \, d\theta - 2\int_{\frac{7\pi}{6}}^{\frac{3\pi}{2}} \frac{1}{2}(3)^2 \, d\theta$$

[pg. 683 #24]
Find the area of the region that is inside $r = 1 - \sin\theta$ and outside $r = 1$

Solution:

Intersection:

$$1-\sin\theta=1$$
$$\sin\theta=0$$
$$\theta=0,\pi,2\pi$$

$$Area=\int_{\pi}^{2\pi}\frac{1}{2}(1-\sin\theta)^2\,d\theta-\int_{\pi}^{2\pi}\frac{1}{2}(1)^2\,d\theta$$
$$=\int_{\pi}^{2\pi}\frac{1}{2}\left(1-2\sin\theta+\sin^2\theta-1\right)d\theta$$
$$=\frac{1}{2}\int_{\pi}^{2\pi}\sin^2\theta-2\sin\theta\,d\theta$$
$$=\frac{1}{2}\int_{\pi}^{2\pi}\frac{1-\cos 2\theta}{2}-2\sin\theta\,d\theta$$
$$=\int_{\pi}^{2\pi}\frac{1}{4}-\frac{1}{4}\cos 2\theta-\sin\theta\,d\theta$$
$$=\frac{1}{4}\theta-\frac{1}{4}\frac{\sin 2\theta}{2}+\cos\theta\Big|_{\pi}^{2\pi}$$
$$=\frac{1}{4}(2\pi)-\frac{1}{4}(0)+1-\left[\frac{1}{4}\pi-0+(-1)\right]$$
$$=\frac{1}{4}\pi+2$$

[pg. 683 #32]

Find the area of the region that is inside $r^2=2\sin 2\theta$ and inside $r=1$

Solution:

“sketch”

Intersection:

$$\mathrm{r}^2 = r^2$$

$$2\sin 2\theta = 1^2$$

$$\sin 2\theta = \frac{1}{2}$$

$$2\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \ldots \qquad \therefore \theta = \frac{\pi}{12}, \frac{5\pi}{12}, \ldots$$

$$Area = 4\int_0^{\frac{\pi}{12}} \frac{1}{2}\left(2\sin 2\theta\right) d\theta + 2\int_{\frac{\pi}{12}}^{\frac{5\pi}{12}} \frac{1}{2}(1)^2 \, d\theta$$

$$= 4\left[-\frac{\cos 2\theta}{2}\right]_0^{\frac{\pi}{12}} + \theta\Big|_{\frac{\pi}{12}}^{\frac{5\pi}{12}}$$

$$= 2\left[-\frac{\sqrt{3}}{2} + 1\right] + \frac{5\pi}{12} - \frac{\pi}{12}$$

$$= -\sqrt{3} + 2 + \frac{\pi}{3}$$

"Lost" points

Example #2 and Example #3 (pg.681, 682) illustrate the importance of sketching.

Example #2 Find the area of the region that lies inside the circle $r = 3\sin\theta$ and outside the cardioid $r = 1 + \sin\theta$.

"graph"

Intersection:

$3\sin\theta = 1 + \sin\theta$

$2\sin\theta = 1$

$\sin\theta = \frac{1}{2}$

$\theta = \frac{\pi}{6}, \frac{5\pi}{6}$

$$area = 2\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{1}{2}\left[(3\sin\theta)^2 - (1+\sin\theta)^2\right] d\theta$$

What happened to the origin??

Example: #3 Find all points of intersection of the curves $r = \cos 2\theta$ and $r = \frac{1}{2}$.

"sketch"

Intersection :

$\cos 2\theta = \frac{1}{2}$

$2\theta = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3}$

$\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$

What happened to the intersection points around the *y*-axis?

Summary of Area Formulae:

Cartesian: Area = Upper Curve – Lower Curve

$$A=\int_a^b\left[f(x)-g(x)\right]dx \text{ or } A=\int_c^d\left[h(y)-i(y)\right]dy$$

Parametric:

$$A=\int_{t_1}^{t_2} y\frac{dx}{dt}dt \text{ or } A=\int_{T_1}^{T_2} x\frac{dy}{dt}dt$$

If a simple closed curve then clockwise $\Rightarrow +$ and counterclockwise $\Rightarrow -$

Polar:

$$A=\frac{1}{2}\int_{\theta_1}^{\theta_2} r^2\,d\theta$$