The Chain Rule (1D)

$$\frac{d}{dt} f(g(t)) = f'(g(t))\, g'(t)$$

- $y = f(x)$ $\quad x = g(t)$ then

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$$

$z = f(x,y)$ $\quad x = g(t)$ $\quad y = h(t)$

We require f to be <u>differentiable</u> as a function of 2 variables

$$\lim_{(h_1,h_2)\to(0,0)} \left| f(x+h_1, y+h_2) - [f_x(x,y)h_1 + f_y(x,y)h_2 + f(x,y)] \right| / (h_1^2+h_2^2)^{1/2} = 0.$$

Then f has a tangent plane at (x,y) approximating f

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

Ex. $z = f(x,y) = x^3 y^5 \qquad x = \cos(t) \qquad y = \sin(t)$

$$\frac{dx}{dt} = -\sin(t) \qquad dy/dt = \cos(t)$$

$$\frac{dz}{dt} = \left(\frac{\partial z}{\partial x}\right)\left(\frac{dx}{dt}\right) + \left(\frac{\partial z}{\partial y}\right)\left(\frac{dy}{dt}\right)$$

$$= 3x^2y^5(-\sin(t)) + 5x^3y^4(\cos(t))$$

$$= -3\cos^2 t \sin^6 t + 5\cos^4 t \sin^4 t$$

2D $\quad z = f(x,y) \qquad x = g(s,t) \qquad y = h(s,t)$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

$U(x_1, \ldots, x_n)$ and $x_i(t_1, t_2, \ldots, t_m)$

$$\frac{\partial U}{\partial t_i} = \sum_{j=1}^{n} \frac{\partial U}{\partial x_j} \frac{\partial x_j}{\partial t_i} \qquad j = 1, 2, \ldots, m$$

$$\begin{bmatrix} \frac{\partial U}{\partial t_1} & \cdots & \frac{\partial U}{\partial t_m} \end{bmatrix} = \begin{bmatrix} \frac{\partial U}{\partial x_1} & \cdots & \frac{\partial U}{\partial x_n} \end{bmatrix} \begin{bmatrix} \frac{\partial x_1}{\partial t_1} & \cdots & \frac{\partial x_1}{\partial t_m} \\ \vdots & \ddots & \\ \frac{\partial x_n}{\partial t_1} & & \frac{\partial x_n}{\partial t_m} \end{bmatrix}$$

Ex $U(s,t) = f(s^2 - t^2, t^2 - s^2)$

show U satisfies

$$t \frac{\partial U}{\partial s} + s \frac{\partial U}{\partial t} = 0$$

① $\frac{\partial U}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} = \frac{\partial f}{\partial x} \cdot 2s + \frac{\partial f}{\partial y} \cdot (-2s)$

② $\frac{\partial U}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} = \frac{\partial f}{\partial x}(-2t) + \frac{\partial f}{\partial y}(2t)$

$t \cdot ① + s \cdot ② = 0 \quad \therefore$ done

Implicit differentiation

The equation $F(x,y)=C$

$$\frac{d}{dx}\, F(x, y(x)) = \frac{d}{dx}\, C = 0$$

$$\frac{\partial F}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial F}{\partial y}\cdot\frac{\partial y}{\partial x} = 0$$

$$\frac{\partial y}{\partial x} = \frac{-\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = \frac{-F_x}{F_y}$$

$$F(x,y) = y^5 - 3x^2y^3 + x^5 = 0$$

$$\frac{\partial y}{\partial x} = -\frac{F_x}{F_y} = \frac{-(-6xy^3 + 5x^4)}{5y^4 - 9x^2y^2}$$

$$F(x,y,z)=C \qquad z = z(x,y)$$

$$\frac{\partial}{\partial x}\, F(x,y,z) = \frac{\partial}{\partial x}\, C = 0$$

$$\frac{\partial F}{\partial x}\underbrace{\frac{\partial x}{\partial x}}_{=1} + \frac{\partial F}{\partial y}\underbrace{\frac{\partial y}{\partial x}}_{=0} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0$$

$$\frac{\partial z}{\partial x} = \frac{-\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} = -F_x / F_z$$

$F(x,y,z) = \sin(x+y) + \sin(x+z) + \sin(y+z) = ($

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = \frac{-\cos(x+y) - \cos(x+z)}{\cos(x+z) + \cos(}$$

Implicit Function Theorem

$F(x,y,z)$ function of 3 variables

$F(a,b,c) = k$ and $F_z(a,b,c) \neq 0$.

Assume F_x, F_y, F_z are continuous near (a,b,c).

<u>Then</u> there exist a differentiable

$z(x,y)$ defined near (a,b) such that

$F(x,y,z(x,y)) = k$ for all (x,y) near (a,b).