Definite Integrals

Example: Find the area between $f(x) = x^2$ and the *x*-axis and the line *x* = 3.

Solution:

$$A(x) = \int f(x)\,dx = \int x^2 dx$$

$$A(x) = \frac{x^3}{3} + C$$

But $A(0)$ is obviously 0.

$$\therefore\ 0 = \frac{0^3}{3} + C \quad \text{or} \quad C = 0$$

$$A(3) = \frac{3^3}{3} = \frac{27}{3} = 9$$

Example: Find the area between $f(x) = x^2$ and the *x*-axis
between the lines *x* = 2 and *x* = 3.

Solution:

$$A(x) = \frac{x^3}{3} + 0 = \frac{x^3}{3}$$

As above $\therefore\ A(3) = \frac{3^3}{3} = 9$ and $A(2) = \frac{2^3}{3} = \frac{8}{3}$

$\therefore$ area between $x = 2$ and $x = 3$ is:

$$9 - \frac{8}{3} = \frac{27 - 8}{3} = \frac{19}{3}$$

<u>A more convenient notation is used</u>:

$$\int_2^3 x^2 dx = \left.\frac{x^3}{3}\right|_2^3 \quad \leftarrow \text{ means } \frac{3^3}{3} - \frac{2^3}{3} = 9 - \frac{8}{3} = \frac{19}{3}$$

Note: If a different antiderivative had been used the answer would have been the same. For example, using

$\frac{x^3}{3}+5$ would produce $\left(\frac{x^3}{3}+5\right)\Bigg|_2^3 = \left(\frac{27}{3}+5\right)-\left(\frac{8}{3}+5\right)=\frac{19}{3}$

The result is the same since you are only interested in the difference and the 5's cancel each other.

N.B. Use the <u>simplest</u> antiderivative. ie. The one with $C = 0$.

The integral symbolism comes from adding all the little rectangles← called a <u>Reimann Sum</u>.

Area from a to b is

$\sum(\text{ all the litle rectangles})$

$$= \lim_{\Delta x \to 0} \sum_{i=1}^{n} f\left(x_i^*\right)\Delta x = \int_a^b f(x)\,dx \leftarrow$$ <u>Definite Integral</u>

Example: Find the area between $g(x) = x^3$ and the x-axis from $x = 1$ to $x = 3$.

Solution:

$$\text{area} = \int_1^3 x^3 dx = \frac{x^4}{4}\Bigg|_1^3 = \frac{3^4}{4}-\frac{1^4}{4} = \frac{81}{4}-\frac{1}{4} = \frac{80}{4} = 20$$

Example: Find the area between $g(x) = -x^3$ and the x-axis from $x = 1$ to $x = 3$.

Solution:

$$\text{area} = \int_1^3 \left(-x^3\right) dx = -\frac{x^4}{4}\Bigg|_1^3 = -\frac{3^4}{4}-\left(-\frac{1^4}{4}\right) = -\frac{81}{4}+\frac{1}{4} = -\frac{80}{4} = -20$$

Something is wrong!

The definite integral represents the area only when it is positive!

$\therefore \int_1^3 \left(-x^3\right) dx = -20$ but the area $= \left|\int_1^3 \left(-x^3\right) dx\right| = |-20| = 20$

<u>Definition of area by integration</u>

$$\text{area} = \int_a^b \left| f(x) \right| dx$$

$$\text{area of } A = \int_a^b f(x)\,dx$$

$$\text{area of } B = -\int_c^d f(x)\,dx$$

Therefore,

$$\int_a^c f(x)\,dx = A - B$$

To find the total area use $\left| f(x) \right|$

$$\text{area} = A + B = \int_a^c \left| f(x)\,dx \right|$$

<u>Further Properties of definite integrals</u>: [Definite integrals are sums so the properties follow the properties of sigma notation.]

1. $\int_a^a f(x)\,dx =$ ____________
2. If $f(x)$ is an odd function then $\int_{-a}^a f(x)\,dx =$ __________
3. If $f(x)$ is an even function then $\int_{-a}^a f(x)\,dx =$ __________.
4. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

5. Additive: $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

Question: Must c be between a and b?

Note: The function can be a piecewise function.

$$\int_a^d f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx + \int_c^d f(x)\,dx$$

6. Constant multiplier: $\int_a^b Cf(x)\,dx = C\int_a^b f(x)\,dx$
7. Sum function: $\int_a^b (f+g)(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

Let's Practice

1. Find the area between the line $y = 2x + 6$ and the x-axis from $x = -5$ to $x = 15$.

   Solution:

$$\int_{-5}^{15} |2x+6|\,dx = \int_{-5}^{-3} (-2x-6)\,dx + \int_{-3}^{15} (2x+6)\,dx$$
$$= \left(-x^2 - 6x\right)\Big|_{-5}^{-3} + \left(x^2 + 6x\right)\Big|_{-3}^{15}$$
$$= (-9+18) - (-25+30) + (225+90) - (9-18)$$
$$= 328$$

2. Find the area bounded by $y = x^2 - 4$ and the x-axis.

   Solution:

$$\text{area} = -\int_{-2}^{2} \left(x^2 - 4\right) dx$$
$$= -\left[\left(\frac{x^3}{3} - 4x\right)\Big|_{-2}^{2`}\right]$$
$$= -\left[\left(\frac{8}{3} - 8\right) - \left(-\frac{8}{3} + 8\right)\right] = \frac{32}{3}$$

3. Find the area between $y = x^3 + 2x^2 - x - 2$ and the x-axis.

Solution:

$0 = x^2(x+2) - 1(x+2)$

$0 = (x+2)(x^2-1)$

$0 = (x+2)(x-1)(x+1)$

x	-2	-1	1	
y	$-$	$+$	$-$	$+$

area = area of A + area of B

$$= \int_{-2}^{-1} f(x)\,dx + \int_{-1}^{1} \left(-f(x)\right) dx$$

$$= \int_{-2}^{-1} \left(x^3 + 2x^2 - x - 2\right) dx + \int_{-1}^{1} \left(-x^3 - 2x^2 + x + 2\right) dx$$

$$= \left(\frac{x^4}{4} + \frac{2x^3}{3} - \frac{x^2}{2} - 2x\right)\Bigg|_{-2}^{-1} + \left(-\frac{x^4}{4} - \frac{2x^3}{3} + \frac{x^2}{2} + 2x\right)\Bigg|_{-1}^{1}$$

$$= \left(\frac{1}{4} - \frac{2}{3} - \frac{1}{2} + 2\right) - \left(4 - \frac{16}{3} - 2 + 4\right) +$$

$$\left(-\frac{1}{4} - \frac{2}{3} + \frac{1}{2} + 2\right) - \left(-\frac{1}{4} + \frac{2}{3} + \frac{1}{2} - 2\right)$$

4. Find the area between $y = \sin x$, the x-axis, $x = -\dfrac{\pi}{3}$ and $x = \dfrac{\pi}{3}$.

Solution:

Since $y = \sin x$ is odd areaA = areaB

$\therefore$ Area $= 2\int_0^{\frac{\pi}{3}} \sin x dx$

$$= 2(-\cos)\Big|_0^{\frac{\pi}{3}} = 2\left[\cos\frac{\pi}{3} + \cos 0\right]$$

$$= 2\left[-\frac{1}{2} + 1\right] = 1$$

Area between Curves

Find the area between the curves $f(x) = x$ and $g(x) = x^2$.

Solution:

Points of intersection are: "sketch"

$x^2 = x$ $\therefore$ shaded area = area under line

$x^2 - x = 0$ − area under parabola

$x(x-1) = 0$ (from $x = 0$ to $x = 1$)

$x = 0 \quad x = 1$

area $= \int_0^1 x dx - \int_0^1 x^2 dx = \left.\frac{x^2}{2}\right|_0^1 - \left.\frac{x^3}{3}\right|_0^1 = \left(\frac{1}{2} - 0\right) - \left(\frac{1}{3} - 0\right) = \frac{1}{6}$

OR area $= \int_0^1 \left(x - x^2\right) dx = \left.\left(\frac{x^2}{2} - \frac{x^3}{3}\right)\right|_0^1 = \left(\frac{1}{2} - \frac{1}{3}\right) - 0 = \frac{1}{6}$

In general

"sketch"

area of $A = \int_a^b \left(f(x) - g(x)\right) dx$

and area = A + B

$= \int_a^b \left(g(x) - f(x)\right) dx + \int_b^c \left(f(x) - g(x)\right) dx$

or $= \int_a^c \left|g(x) - f(x)\right| dx$

1. Find the area bounded by the curves $y = x^2 - 2x$ and $y = 6x - x^2$.

Solution:

Intersection points are:

$$x^2 - 2x = 6x - x^2$$

$$2x^2 - 8x = 0$$

$$2x(x-4) = 0$$

$$x = 0 \quad x = 4$$

$$Area = \int_0^4 \left[\left(6x - x^2\right) - \left(x^2 - 2x\right)\right] dx$$

"sketch"

$$= \int_0^4 \left(8x - 2x^2\right) dx$$

$$= \left(4x^2 - \frac{2x^3}{3}\right)\Bigg|_0^4 = \frac{64}{3}$$

2. Find the area between $y = x$ and $y = \sqrt[3]{x}$

Solution: "sketch"

Intersection points are:

$$x = \sqrt[3]{x}$$

$$x^3 = x$$

$$x^3 - x = 0$$

$$x(x-1)(x+1) = 0 \qquad \therefore x = 0, \pm 1$$

$$\text{area} = \int_{-1}^{0} \left(x - \sqrt[3]{x}\right) dx + \int_{0}^{-1} \left(\sqrt[3]{x} - x\right) dx$$

$$= \left(\frac{x^2}{2} - \frac{3x^{\frac{4}{3}}}{x}\right)\Bigg|_{-1}^{0} + \left(\frac{3x^{\frac{4}{3}}}{4} - \frac{x^2}{2}\right)\Bigg|_{0}^{1}$$

$$= 0 - \left(\frac{1}{2} - \frac{3}{4}\right) + \left(\frac{3}{4} - \frac{1}{2}\right) - 0 = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

<u>OR</u>

Since both functions are odd you can use symmetry.

$$\text{area} = 2\int_0^1 \left(\sqrt[3]{x} - x\right) dx == 2\left(\left(\frac{3}{4}x^{\frac{4}{3}} - \frac{x^2}{2}\right)\Bigg|_0^1\right) = 2\left(\frac{3}{4} - \frac{1}{4}\right) = 2\left(\frac{1}{4}\right) = \frac{1}{2}$$

3. Find the area between $y^2 = 2x - 2$ and $y = x - 5$.

   Solution: "sketch"

   intersection points are:

$$\frac{y^2+2}{2} = y+5$$

$$y^2 + 2 = 2y + 10$$

$$y^2 - 2y - 8 = 0$$

$$(y-4)(y+2) = 0$$

$$y = 4 \quad y = -2$$

$$x = 9 \quad x = 3$$

Method 1

$$\text{area} = \int_1^3 \left(\sqrt{2x-2} - \left(-\sqrt{2x-2}\right)\right) dx + \int_3^9 \left(\sqrt{2x-2} - (x-5)\right) dx$$

$$= \int_1^3 \left(2\sqrt{2x-2}\right) dx + \int_3^9 \left(\sqrt{2x-2} - x + 5\right) dx$$

Method 2 (Much easier) *x* and *y* change roles as you rotate the diagram 90^0.

$$\text{area} = \int_{-2}^{4}\left[(y+5)-\left(\frac{y^2+2}{2}\right)\right]dy$$

$$= \int_{-2}^{4}\left(y+5-\frac{y^2}{2}-1\right)dy$$

$$= \int_{-2}^{4}\left(y-\frac{y^2}{2}+4\right)dy$$

$$= \left(\frac{y^2}{2}-\frac{y^3}{6}+4y\right)\Bigg|_{-2}^{4}$$

$$= \left(8-\frac{32}{3}+16\right)-\left(2+\frac{4}{3}-8\right)$$

$$= \left(24-\frac{32}{3}\right)-\left(-6+\frac{4}{3}\right) = 30-12 = 18$$

Homework:

4. Find the area between the curves:

   $y = x^3 + x + 1$ and $y = x^2 + x + 1$ from $x = -1$ to $x = 2$

5. Find the area between the curves:

   $y = x^2 - 2x$ and $y = -x^2 + 2x + 6$