Integration by Parts

We integrate a product rule.

$(f \cdot g)' = f'g + fg'$ ← product rule

Integrating both sides we get

$\int (f \cdot g)' = \int f'g + \int fg'$

Hence $f \cdot g = \int f'g + \int fg'$ ← is a sum of two integrals

Transposing either integral we get:

$$f \cdot g - \int f'g = \int fg' \quad \text{or} \quad \boxed{\int fg' = f \cdot g - \int f'g} \leftarrow \underline{\text{Integration by Parts}}$$

Usually written as: $\boxed{\int u\, dv = uv - \int v\, du}$

Thus an integral is change to a product of two functions $f \cdot g$ ← no problem and a different integral $\int f'g$ ← which better be <u>easier</u> than the original integral $\int fg'$!!!!!!

Question: Which of the two integrals $\int x \sin x\, dx$ and $\int \cos x\, dx$ is easier?

Integration by parts will change the problem of integrating $\int x \sin x\, dx$ into integrating $\int \cos x\, dx$.

Example: $\int x \sin x\, dx$ ← I wish the x was not there

Solution:

$$\begin{aligned} &Let\ u = x && dv = \sin x\, dx \\ &\quad du = dx && v = -\cos x \end{aligned}$$

$$\int x \sin x\, d = -x \cos x - \int -\cos x d = -x \cos x + \int \cos x\, d = -x \cos x + \sin x + C$$

It is very important what you select to be u and what you select to be dv!!!!!

Integration by parts may be <u>repeated</u>

Example: $\int x^2 \sin x\, dx$ ← I wish the x^2 was not there.

Solution:

$$\begin{aligned} &Let\ u = x^2 \qquad dv = \sin x\, dx \\ &\quad du = 2x dx \qquad v = -\cos x \end{aligned}$$

$$I = \int x^2 \sin x\, dx = -x^2 \cos x - \int -2x \cos x\, dx = -x^2 \cos x + 2\int x \cos x\, dx$$

$$\begin{aligned} &Let\ u = x \qquad dv = \cos x\, dx \\ &\quad du = dx \qquad v = \sin x \end{aligned} \quad \leftarrow \text{second time}$$

$$I = -x^2 \cos x + 2\left[x \sin x - \int \sin x\, dx\right] = -x^2 \cos x + 2x \sin x + 2\cos x + C$$

If the question was $\int x^3 \sin x\, dx$ we would use integration by parts three times or resort to a <u>reduction formula</u>.

Another type of question is a "cycler". In this case neither term disappears.

Example: Find $\int e^x \sin x dx$ ←neither the exponential nor the trig function will disappear on differentiation.

Solution:

$$\begin{aligned} &Let\, u = e^x \ \ and \ \ dv = \sin x\, dx \\ &\quad du = e^x dx \qquad v = -\cos x \end{aligned}$$

$$I = \int e^x \sin x dx = -e^x \cos x - \int -e^x \cos x\, dx = -e^x \cos x + \int e^x \cos x\, dx$$

$$\begin{aligned} &Let\, u = e^x \ \ and \ \ dv = \cos x\, dx \\ &\quad du = e^x dx \qquad v = \sin x \end{aligned}$$

$$I = -e^x \cos x + e^x \sin x - \int e^x \sin x dx + C = -e^x \cos x + e^x \sin x - I + C$$

$$2I = -e^x \cos x + e^x \sin x + C$$

$$I = \int e^x \sin x dx = \frac{-e^x \cos x + e^x \sin x}{2} + K \qquad \leftarrow \text{ where } K = \frac{C}{2}$$

OR

$$Let\,u = \sin x \;\; and \;\; dv = e^x\,dx$$

$$du = \cos x dx \qquad v = e^x$$

$$I = \int e^x \sin x dx = e^x \sin x - \int e^x \cos x\, dx$$

$$Let\,u = \cos x \;\; and \;\; dv = e^x\,dx$$

$$du = -\sin x dx \qquad v = e^x$$

$$I = e^x \sin x - e^x \cos x + \int e^x \sin x dx + C = -e^x \cos x + e^x \sin x - I + C$$

$$2I = -e^x \cos x + e^x \sin x + C$$

$$I = \int e^x \sin x dx = \frac{-e^x \cos x + e^x \sin x}{2} + K \qquad \leftarrow \text{ where } K = \frac{C}{2}$$

Example: Find $\int e^{3x} \cos x dx \leftarrow$ another cycler

Solution:

$$Let\;\; u = e^{3x} \qquad dv = \cos x\, dx$$

$$du = 3e^{3x}dx \qquad v = \sin x$$

$$I = \int e^{3x} \cos x\, dx = e^{3x} \sin x - 3\int e^{3x} \sin x\, dx$$

$$Let\;\; u = e^{3x} \qquad dv = \sin x\, dx$$

$$du = 3e^{3x}dx \qquad v = -\cos x$$

$$I = e^{3x} \sin x - 3\left[-e^{3x} \cos x - 3\int -e^{3x} \cos x\, dx\right] = e^{3x} \sin x + 3e^{3x} \cos x - 9I + C$$

$$10I = e^{3x} \sin x + 3e^{3x} \cos x + C$$

$$I = e^{3x} \cos x\, dx = \frac{e^{3x} \sin x + 3e^{3x} \cos x}{10} + K \qquad \leftarrow Where\; K = \frac{C}{10}$$

The last type I will mention is the <u>inverse trig or logarithmic function</u>. Since you only know how to differentiate these functions you should let *u*=this function.

Example: Find $\int \ln(2x)\,dx$

Solution:

We cannot let dv = ln(2x) since we do not know its integral. In fact this is the question we are being asked!
Therefore,

$$Let\ u = \ln(2x) \qquad dv = dx$$

$$du = \frac{1}{2x}2dx = \frac{1}{x}dx \qquad v = x$$

$$\int \ln(2x)\,dx = x\ln(2x) - \int dx = x\ln(2x) - x + C$$

Similarly
Example: Find $\int \sin^{-1} x\,dx$
Solution:
We only know the derivative of $\sin^{-1}x$.
Therefore,

$$Let\ u = \sin^{-1} x \qquad dv = dx$$

$$du = \frac{1}{\sqrt{1-x^2}}dx \qquad v = x$$

$$\int \sin^{-1} x\,dx = x\sin^{-1} x - \int \frac{x}{\sqrt{1-x^2}}dx$$

$$= x\sin^{-1} x + \left(1-x^2\right)^{\frac{1}{2}} + C$$

When should you use integration by parts? Whenever you cannot evaluate the given integral by using integration by parts will change it to an easier integral. Once you decide on using integration by parts, how do you decide what should be u and what should be dv? Practice, which generates experience is invaluable. However, here are some suggestions.

Type	Suggestion
1. $\int (polynomial)(trig)\,dx$ Or $\int (polynomial)(\text{exp}onential)\,dx$	Choose u to be a power of x to lower its exponent.

2. $\int (trig)(\text{exp}onential)\,dx$ Choose u to be either one. This is a cycler. Just stay with the same u throughout.

3. $\int (polynomial)(inverse\,trig)\,dx$ Choose u to be the inverse trig function.

$\int (polynomial)(\text{log}arithmic)\,dx$ Choose u to be the logarithmic function.

These suggestions are not foolproof but they will work most of the time.