Inverse Trigonometric Functions

High school review:

A Questions without inverse functional notation:

Example1: Find $\tan\theta$ if $\sin\theta = \frac{1}{2}$ and $\theta \in I$

Solution:

"sketch a unit circle or a right triangle"

(a) Coordinates of the point on unit circle are $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$

$$\therefore \tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{3}}$$

OR

(b) $\tan\theta = \frac{opposite}{adjacent} = \frac{1}{\sqrt{3}}$

Example2: Find $\cot\theta$ if $\cos\theta = -\frac{\sqrt{3}}{2}$ and $\theta \in II$

Solution:

"sketch a unit circle or a right triangle"

(a) Coordinates of the point on unit circle are $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$

$$\therefore \cot\theta = \frac{\cos\theta}{\sin\theta} = -\frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = -\sqrt{3}$$

OR

(b) $\cot\theta = \frac{adjacent}{opposite} = -\frac{\sqrt{3}}{1}$

Example2: Find $\sin\theta$ if $\tan\theta = -1$ and $\theta \in IV$

Solution:

"sketch a unit circle or a right triangle"

(a) Coordinates of the point on unit circle are $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$

$$\therefore \sin\theta = -\frac{1}{\sqrt{2}}$$

OR

(b) $\sin\theta = \dfrac{opposite}{hypotenuse} = -\dfrac{1}{\sqrt{2}}$

B. The above three example can be stated more concisely by using inverse trig functional notation. Each question would be stated as follows:

$(a)\ \tan\left(\sin^{-1}\frac{1}{2}\right) = \frac{1}{\sqrt{3}}$ $\quad (b)\ \cot\left(\cos^{-1}\frac{-\sqrt{3}}{2}\right) = -\sqrt{3}$ $\quad (c)\ \sin\left(\tan^{-1}(-1)\right) = -\frac{\sqrt{3}}{2}$

{Note: we can also use arcsin etc. in place of $\sin^{-1}$. It emphasizes that the value is an arc!}

Therefore,

Example 4: Find the value of $\cos\left(\sin^{-1}\frac{1}{2}\right)$

means " Find the value of $\cos\theta$ if $\sin\theta = \frac{1}{2}$ and $\theta \in I$ "

Example 5: Find the value of $\tan\left(\cos^{-1}\left(-\frac{1}{2}\right)\right)$

means " Find the value of $\tan\theta$ if $\cos\theta = -\frac{1}{2}$ and $\theta \in II$ "

Example 4: Find the value of $\sec\left(\sin^{-1}\left(-\frac{3}{5}\right)\right)$

means " Find the value of $\sec\theta$ if $\sin\theta = -\frac{3}{5}$ and $\theta \in IV$ "

<u>Exercise to review more notation and methods</u>

Find the exact value of:

$$\sin^{-1}\left(\sin\frac{5\pi}{6}\right)$$

1. $\sin\left(\sin^{-1}\frac{3}{4}\right)$
2. $\tan\left(\tan^{-1}3\right)$
3. $\sin^{-1}\left(\sin\frac{\pi}{6}\right)$
4. $\tan^{-1}\left(\tan\frac{5\pi}{4}\right)$
5. $\sin\left(\tan^{-1}\frac{4}{3}\right)$
6. $\tan\left(\sin^{-1}\frac{\sqrt{5}}{5}\right)$
7. $\cos\left(\tan^{-1}(-2)\right)$
8. $\tan\left(\sin^{-1}\frac{4}{5}\right)$
9. $\sin\left(2\sin^{-1}\frac{\sqrt{3}}{2}\right)$
10. $\sec\left(\cos^{-1}\frac{7}{10}\right)$
11. $\cos\left(2\tan^{-1}\left(-\frac{5}{12}\right)\right)$

Be sure you know the definitions and graphs of the inverse trig functions!
The definitions and the graphs are found on pg. 72 – 74 in Stewart.

The <u>three basic ones are much more important than the remaining ones.</u>

<u>Definition of $\sin^{-1}x$</u>

$\sin^{-1}x = y$ provided $\sin y = x$ and $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$ [ie. $y \in I \text{ or } IV$]

Remember!! y is an arc length (or an angle).

"sketches" Range of $\sin^{-1}x = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and its domain is $[-1,1]$

<u>Definition of $\tan^{-1}x$</u>

$\tan^{-1}x = y$ provided $\tan y = x$ and $-\frac{\pi}{2} < y < \frac{\pi}{2}$ [ie. $y \in I \text{ or } IV$]

Remember!! y is an arc length (or an angle).

"sketches" Range of $\tan^{-1}x = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and its domain is R

<u>Definition of $\cos^{-1}x$</u>

$\cos^{-1}x = y$ provided $\cos y = x$ and $0<y<\pi$ [ie. $y \in I \text{ or } II$]

Remember!! y is an arc length (or an angle).

"sketches" Range of $\cos^{-1}x = [0, \pi]$ and its domain is $[-1,1]$

Try something cute. Sketch each of the following:

(a) $y = \sin^{-1}x + \frac{\pi}{2}$ (b) $y = 2\tan^{-1}x$ (c) $y = \cos^{-1}\left(\frac{1}{2}x\right)$

Just as in grade twelve these are functions and behave as any other transformed functions.

<u>Non-numerical type of questions</u>

Simplify

1. $\sin\left(\cos^{-1}x\right)$
2. $\cos\left(\sin^{-1}x\right)$
3. $\cos\left(\tan^{-1}x\right)$

4. $\sin\left(\tan^{-1} x\right)$
5. $\tan\left(\cos^{-1} x\right)$
6. $\tan\left(\sec^{-1} x\right)$

Derivatives of Inverse Trigonometric Functions

[You are responsible for the proofs of these theorems. The proofs are easy!]

Theorem: If $y = \sin^{-1} x$ then $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{1-x^2}}$

Proof:

$$y = \sin^{-1} x$$

$$\textit{Hence, } \sin y = x \qquad \text{"}\textit{sketch}\text{"}$$

$$\cos y \frac{dy}{dx} = 1 \qquad 1^2 = a^2 + x^2$$

$$\frac{dy}{dx} = \frac{1}{\cos y} \qquad \therefore a = \pm\sqrt{1-x^2}$$

$$= \frac{1}{\dfrac{\sqrt{1-x^2}}{1}} \qquad \textit{But} -\frac{\pi}{2} \le y \le \frac{\pi}{2}$$

$$= \frac{1}{\sqrt{1-x^2}} \qquad \therefore a = \sqrt{1-x^2}$$

$\textit{Since } \cos y \ge 0 \textit{ in quadrants I and IV}.$

Theorem: If $y=\tan^{-1}x$ then $\dfrac{dy}{dx}=\dfrac{1}{1+x^2}$

Proof:

$$y=\tan^{-1}x$$

$$Hence,\ \tan y = x$$

$$\sec^2 y\frac{dy}{dx}=1$$

$$\frac{dy}{dx}=\frac{1}{\sec^2 y}$$

$$=\frac{1}{1+\tan^2 y}$$

$$=\frac{1}{1+x^2}$$

Theorem: If $y=\cos^{-1}x$ then $\dfrac{dy}{dx}=-\dfrac{1}{\sqrt{1-x^2}}$

Proof:

$$y=\cos^{-1}x$$

$$Hence,\ \cos y = x \qquad \text{"}sketch\text{"}$$

$$-\sin y\frac{dy}{dx}=1 \qquad 1^2=a^2+x^2$$

$$\frac{dy}{dx}=\frac{1}{-\sin y} \qquad \therefore a=\pm\sqrt{1-x^2}$$

$$=\frac{1}{\frac{-\sqrt{1-x^2}}{1}} \qquad But\ 0\le y\le\pi$$

$$=-\frac{1}{\sqrt{1-x^2}} \qquad \therefore a=\sqrt{1-x^2}$$

$$Since\ \sin y\ge 0\ in\ quadrants\ I\ and\ II.$$

Be careful of chain rule questions involving inverse trig derivatives. Since

$$\left(\sin^{-1} x\right)' = \frac{1}{\sqrt{1-x^2}} \quad \text{and} \quad \left(\tan^{-1} x\right)' = \frac{1}{1+x^2}$$

with the chain rule become:

$$\left(\sin^{-1} \square\right)' = \frac{1}{\sqrt{1-\square^2}}\square' \quad \text{and} \quad \left(\tan^{-1} \square\right)' = \frac{1}{1+\square^2}\square' \quad etc.$$

Examples: Find y' if :

$(a)\quad y = \sin^{-1}\left(x^2\right) \qquad (b)\quad y = \tan^{-1}\left(2\sin x\right) \qquad (c)\quad \sin\left(x+\cos^{-1} x\right)$

Pg. 234

#44 Differentiate $h(x) = \sqrt{1-x^2}\,(\arcsin x) \qquad \leftarrow \textit{means} \sqrt{1-x^2}\left(\sin^{-1} x\right)$

Solution:

$$h'(x) = \frac{1}{2\sqrt{1-x^2}}(-2x)\arcsin x + \sqrt{1-x^2}\left(\frac{1}{\sqrt{1-x^2}}\right) = \frac{1}{2\sqrt{1-x^2}}(-2x)\arcsin x + 1$$

#46 Differentiate $y = \tan^{-1}\left(x-\sqrt{1+x^2}\right)$

Solution:

$$y' = \frac{1}{1+\left(x-\sqrt{1+x^2}\right)^2}\left[1-\frac{1}{2\sqrt{1+x^2}}(2x)\right]$$

Reversing the differentiation process we also have:

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + C \qquad \text{and} \qquad \int \frac{1}{1+x^2}\,dx = \tan^{-1} x + C \quad etc.$$

Examples:

1. $\int \frac{x}{\sqrt{1-x^4}}\,dx = \int \frac{x}{\sqrt{1-\left(x^2\right)^2}}\,dx = = I$

$$Let\ u = x^2 \qquad du = 2x\,dx$$

$$I = \int \frac{\frac{1}{2}du}{\sqrt{1-u^2}} = \frac{1}{2}\sin^{-1}u + C = \frac{1}{2}\sin^{-1}\left(x^2\right) + C$$

2. (pg. 421 #22)

$$\int \frac{\tan^{-1}x}{1+x^2}\,dx = \int \tan^{-1}x\left(\frac{1}{1+x^2}\right)dx = \frac{\left(\tan^{-1}x\right)^2}{2} + C$$

OR

$$Let\ u = \tan^{-1}x \quad du = \frac{1}{1+x^2}\,dx$$

$$\int \frac{\tan^{-1}x}{1+x^2}\,dx = \int u\,du = \frac{u^2}{2} + C = \frac{\left(\tan^{-1}x\right)^2}{2} + C$$

3. (pg. 421#66)

$$I = \int_0^{\frac{1}{2}} \frac{\sin^{-1}x}{\sqrt{1-x^2}}\,dx = \left.\frac{\left(\sin^{-1}x\right)^2}{2}\right|_0^{\frac{1}{2}} = \frac{\left(\sin^{-1}\left(\frac{1}{2}\right)\right)^2}{2} - \frac{\left(\sin^{-1}0\right)^2}{2}$$

$$= \frac{\left(\frac{\pi}{6}\right)^2}{2} - \frac{0^2}{2} = \frac{\pi^2}{72}$$

3. (pg. 421#66)

OR

$$Let\ u = \sin^{-1} x \quad du = \frac{1}{\sqrt{1-x^2}}dx$$

x	u
$\frac{1}{2}$	$\frac{\pi}{6}$
0	0

$$I = \int_0^{\frac{\pi}{6}} u\,du = \left.\frac{u^2}{2}\right|_0^{\frac{\pi}{6}} = \frac{\left(\frac{\pi}{6}\right)^2}{2} = \frac{\pi^2}{72}$$

Furthermore

1. $$\int \frac{\left(\tan^{-1} x\right)^3}{2+2x^2}dx = \frac{1}{2}\int \left(\tan^{-1} x\right)^3 \cdot \frac{1}{1+x^2}dx = \frac{1}{2}\frac{\left(\tan^{-1} x\right)^4}{4} + C$$

OR

$$Let\ u = \tan^{-1} x \qquad du = \frac{1}{1+x^2}dx$$

$$\int \frac{\left(\tan^{-1} x\right)^3}{2+2x^2}dx = \frac{1}{2}\int u^3\,du = \frac{1}{2}\cdot\frac{u^4}{4} + C = \frac{1}{8}\left(\tan^{-1} x\right)^4 + C$$

2. $$\int \frac{4}{\sqrt{1-x^2}\,\sin^{-1} x}dx = 4\int \frac{\frac{1}{\sqrt{1-x^2}}}{\sin^{-1} x}dx = 4\ln\left|\sin^{-1} x\right| + C$$

OR

$$Let\ u = \sin^{-1} x \quad du = \frac{1}{\sqrt{1-x^2}}dx$$

$$\int \frac{4}{\sqrt{1-x^2}\,\sin^{-1} x}dx = 4\int \frac{du}{u} = 4\ln|u| + C = 4\ln\left|\sin^{-1} x\right| + C$$

Homework

3. Find $\int_0^{\sqrt{3}} \frac{\sqrt{\tan^{-1} x}}{3+3x^2}dx$