L'Hopital's Rule

Key Previous Knowledge:

1. Fact: $e^{\ln y} = y$
2. Logarithmic Differentiation

Find $\frac{dy}{dx}$ if $y = (\sin x)^x$

Key first step was to take the logarithm of both side.

Hence, $\ln y = x \ln(sinx)$ etc.

Form One:

If $f(a) = g(a) = 0$ and $f'(a)$ and $g'(a)$ exist with $g'(a) \neq 0$

then $\lim\limits_{x\to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}$.

Examples:

Find: (a) $\lim\limits_{x\to 0} \frac{3x - \sin x}{x}$ (b) $\lim\limits_{x\to 0} \frac{\sqrt{1+x} - 1}{x}$

<u>Algebraic Justification</u>:

Method: Work backwards from $f'(a)$ and $g'(a)$

$$\frac{f'(a)}{g'(a)} = \frac{\lim\limits_{x\to a} \frac{f(x) - f(a)}{x-a}}{\lim\limits_{x\to a} \frac{g(x) - g(a)}{x-a}} = \lim_{x\to a} \frac{\frac{f(x) - f(a)}{x-a}}{\frac{g(x) - g(a)}{x-a}} = \lim_{x\to a} \frac{f(x) - f(a)}{g(x) - g(a)}$$

$$= \lim_{x\to a} \frac{f(x) - 0}{g(x) - 0} = \lim_{x\to a} \frac{f(x)}{g(x)}$$

Graphical Argument:

Zoom in on the graphs of *f* and *g* at *a*. Since $f'(a)$ and $g'(a)$ exists the graphs of *f* and g, near *a*, appear to be straight lines. (Local Linearity) "graph"

If x is very near a then $\dfrac{f(x)}{g(x)} = \dfrac{f(x)\big/ x-a}{g(x)\big/ x-a} = \dfrac{m_f}{m_g}$

As $x \to a$, m_f and m_g approach $f'(a)$ and $g'(a)$.

$\therefore \lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = \lim\limits_{x\to a} \dfrac{m_f}{m_g} = \dfrac{f'(a)}{g'(a)}$

Question: What happens if $g'(a) = 0$?

If $f'(a)$ is also 0 we have a more general L'Hopital Rule.

Form 2:

If f and g are differentiable and $f(a) = g(a) = 0$ then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}. \text{ provided this limit exists.}$$

Example:

$$\lim_{x\to 0} \frac{x^5}{x^3} \qquad \left[\frac{0}{0}\right]$$

$$\underset{=}{[\mathrm{H}]} \lim_{x\to 0} \frac{5x^4}{3x^2} \underset{=}{[\mathrm{H}]} \lim_{x\to 0} \frac{20x^3}{6x} \underset{=}{[\mathrm{H}]} \lim_{x\to 0} \frac{60x^2}{6} = \lim_{x\to 0} 10x^2 = 0$$

Second Basic indeterminant form is $\frac{\infty}{\infty}$. Actually, a form of $\frac{0}{0}$ since $\frac{\infty}{\infty} \approx \frac{\frac{1}{0}}{\frac{1}{0}} \approx \frac{0}{0}$.

Example:

$$\lim_{x\to\infty} \frac{x}{e^x} \qquad \left[\frac{\infty}{\infty}\right]$$

$$\underset{=}{[H]} \lim_{x\to\infty} \frac{1}{e^x} = 0$$

Indeterminant forms: (We do not know the value, nor if it even exists.)
**Basic: (All other forms must be changed to one of these before applying L'Hopital's Rule)

Example	Form
$\lim_{x\to 0} \frac{x}{\sin x}$	$\frac{0}{0}$
$\lim_{x\to\infty} \frac{e^x}{x}$	$\frac{\infty}{\infty}$

Forms requiring some algebraic manipulation to change to one of the Basic forms.

Example	Form	Strategy	New Form
$\lim_{x\to\infty} x\sin\frac{1}{x}$	$\infty\times 0$	$=\lim_{x\to\infty}\frac{\sin\frac{1}{x}}{1/x}$	$\frac{0}{0}$
$\lim_{x\to(\pi/2)^-}(\sec x-\tan x)$	$\infty-\infty$	$=\lim_{x\to(\pi/2)^-}\left(\frac{1-\sin x}{\cos x}\right)$	$\frac{0}{0}$

Forms requiring the following steps: (Logarithmic)

1. Let *y* equal the expression without the limit..
2. Take *ln* of both sides.
3. Find the limit.
4. Change back to *y*. Thus answering the question

Example	Form	Following the steps
$\lim\limits_{x\to 0^+} x^{\sqrt{x}}$	0^0	Let $y = x^{\sqrt{x}}$ $\ln y = \sqrt{x}\ln x$ $\lim\limits_{x\to 0^+}\ln y = \lim\limits_{x\to 0^+}\sqrt{x}\ln x \ [0\times\infty]$ $= \lim\limits_{x\to 0^+}\dfrac{\ln x}{1/\sqrt{x}} \quad \left[\dfrac{\infty}{\infty}\right]$ $\overset{[H]}{=} \lim\limits_{x\to 0^+}\dfrac{1/x}{-\frac{1}{2}x^{-\frac{3}{2}}} = \lim\limits_{x\to 0^+} -2x^{\frac{1}{2}} = 0$

$$\therefore \lim_{x\to 0^+} x^{\sqrt{x}} = \lim_{x\to 0^+} y = \lim_{x\to 0^+} e^{\ln y} = e^{\lim\limits_{x\to 0^+}\ln y} = e^0 = 1$$

$\lim\limits_{x\to 0^+}\left(\dfrac{1}{x}\right)^x$	∞^0	Same method as above
$\lim\limits_{x\to\infty}\left(1+\dfrac{1}{x}\right)^x$	1^∞	Same method as above

This last example is an alternate definition of the number e and is used to change the compound interest formula $A = P\left(1+\frac{i}{n}\right)^{nt}$ into the continuous compounding formual $A = Pe^{rt}$.

Not all forms are indeterminant! We know the values
For example

$$\infty + \infty = \infty \qquad 5\times\infty = \infty \qquad \frac{0}{\infty} = 0$$

$$\frac{3}{\infty} = 0 \qquad \frac{\infty}{3} = \infty \qquad (1.1)^\infty = \infty$$

$$(0.9)^\infty = 0$$

Find each limit, if it exists.

(a) $\lim\limits_{x\to 1}\dfrac{x^3-5x^2+6x-2}{x^5-4x^4+7x\ \ -9x+5}$

(b) $\lim\limits_{x\to\infty}\dfrac{1-x^3}{4x^3+x+1}$

(c) $\lim\limits_{x\to 0}\dfrac{\sin x}{x^3}$

(d) $\lim\limits_{x\to\infty}\dfrac{e^x}{x^{100}}$

(e) $\lim\limits_{x\to 0^+}\left(\dfrac{1}{x}-\dfrac{1}{\sin x}\right)$

(f) $\lim\limits_{x\to 0^+} x^x$

(g) $\lim\limits_{x\to 0}\dfrac{\tan x-x}{x^3}$

(h) $\lim\limits_{x\to\infty}\left(2+\dfrac{3}{x}\right)^{5x}$