Limits

Let $f(x,y)$ be a function with domain D.

$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$

"The limit of $f(x,y)$ as (x,y) approaches (a,b) is L"

means that we can make $f(x,y)$ as close to L as we want as long as we take (x,y) sufficiently close to (a,b).

Note (a,b) need not be in D.

Ex. Show $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$ does not exist.

approach along the line $y=0$ $\lim_{x\to 0} \frac{x^2}{x^2} = 1$

" " " " $x=0$ $\lim_{y\to 0} \frac{-y^2}{y^2} = -1$

$1 \neq -1$ $\therefore$ The limit does not exist.

$$\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}$$

approach along the line $y=mx$

$$\lim_{x\to 0} \frac{x(mx)^2}{x^2+(mx)^4} = \lim_{x\to 0} \frac{m^2x^3}{x^2+m^4x^4} = \lim_{x\to 0} \frac{m^2x}{1+m^4x^2}$$
$$= 0$$

Ex show that $\lim\limits_{(x,y)\to(0,0)} \dfrac{3x^2y}{x^2+y^2} = 0$ exists.

To actually prove existence we have to use the Squeeze Theorem.

Let $h(x,y)$ and $g(x,y)$ and $f(x,y)$ be functions such that $h(x,y) \le f(x,y) \le g(x,y)$

and $\lim\limits_{(x,y)\to(a,b)} h(x,y) = L = \lim\limits_{(x,y)\to(a,b)} g(x,y)$

Then $\lim\limits_{(x,y)\to(a,b)} f(x,y) = L$

let $h = 0$ let $g = 3|y|$

$$0 \le \frac{3x^2|y|}{x^2+y^2} = 3|y| \frac{x^2}{x^2+y^2} \le 3|y|$$

$\lim\limits_{y\to 0} 0 = 0$ $\lim\limits_{y\to 0} 3|y| = 0$

$\therefore$ $0 \le \lim\limits_{(x,y)\to(0,0)} \dfrac{3x^2|y|}{x^2+y^2} \le 0$

$\therefore$ $\lim\limits_{(x,y)\to(0,0)} \dfrac{3x^2|y|}{x^2+y^2} = 0$

$\therefore$ $\lim\limits_{(x,y)\to(0,0)} \dfrac{3x^2y}{x^2+y^2} = 0.$