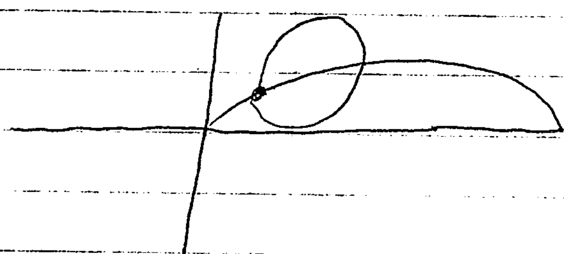
$x(t) = 2t/(1+t^2)$ $\quad$ $y(t) = (-1+t^2)/(1+t^2)$

Typical Trick Question

Generates all P triples

The Cycloid

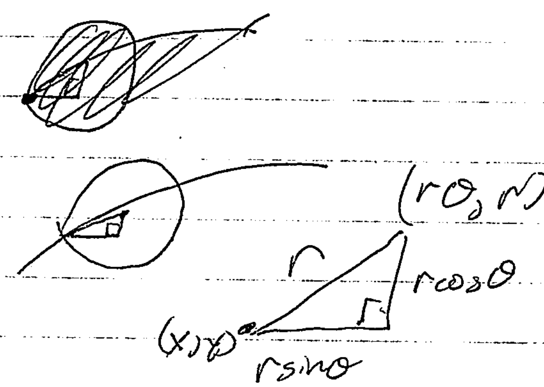


θ = angle of rotation

r = wheel radius

$x = r\theta - r\sin\theta$

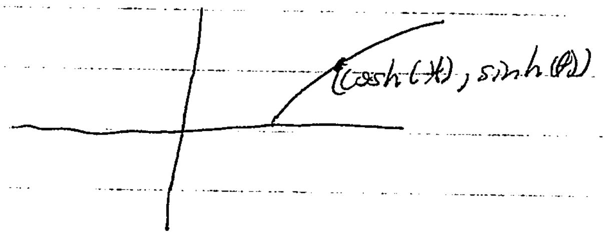
$y = r - r\cos\theta$



hyperbola

$\cosh(t) = \frac{1}{2}(e^{t} + e^{-t})$ $\quad \sinh(t) = \frac{1}{2}(e^{t} - e^{-t})$

$x^2 - y^2 = \frac{1}{4}((e^{2t} + 2 + e^{-2t}) - (e^{2t} - 2 + e^{-2t}))$
$= \frac{1}{4}(4) = 1$



$x = f(t)$, $\quad y = g(t)$ eliminate the parameter to get

$y = F(x)$, where $\quad g(t) = F(f(t))$
$g'(t) = F'(f(t)) \cdot f'(t)$ chain rule

$F'(x) = \frac{g'(t)}{f'(t)}$ assuming $f'(t) \neq 0$.

$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$ $\quad$ " $\quad \frac{dx}{dt} \neq 0$

$x = \sin(t)$ $\quad y = \cos(t)$

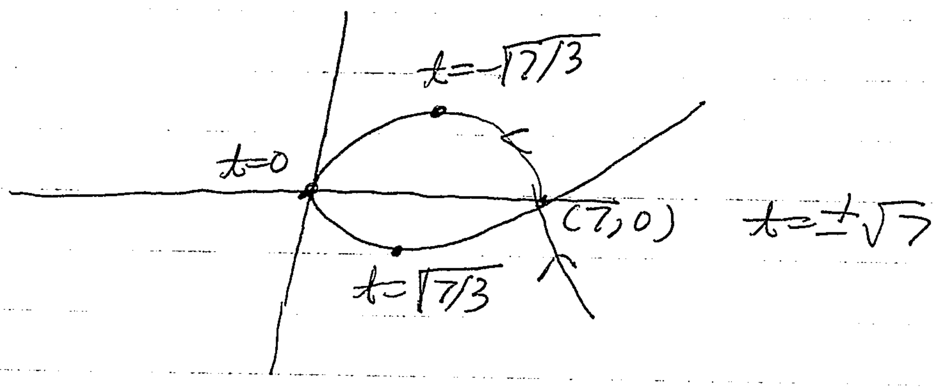
$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-\sin(t)}{\cos(t)} = -\tan(t)$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \qquad \frac{dy}{dt} = 0 \Rightarrow \text{horizontal tangent}$$

$$\frac{dx}{dt} = 0 \Rightarrow \text{vertical} \quad "$$

$$x = t^2 \qquad y = t^3 - 7t = t(t^2-7)$$

$$y^2 = t^2(t^2-7)^2 = x(x-7)^2$$



$$\frac{dy}{dt} = 3t^2 - 7 \qquad t = \pm\sqrt{7/3} \Rightarrow \frac{dy}{dt} = 0$$

$$\frac{dx}{dt} = 2t \qquad t = 0 \Rightarrow \frac{dx}{dt} = 0$$

Second derivatives

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$$

$$\frac{d}{dt}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{dy/dt}{dx/dt}\right) = \frac{(d^2y/dt^2)\cdot(dx/dt) - (d^2x/dt^2)(\ )}{(dx/dt)^2}$$

$$\therefore \frac{d^2y}{dx^2} = \frac{(d^2y/dt^2)(dx/dt) - (d^2x/dt^2)(dy/dt)}{(dx/dt)^3}$$

Ex $\quad x = t^2 \quad y = t^3 - 7t$

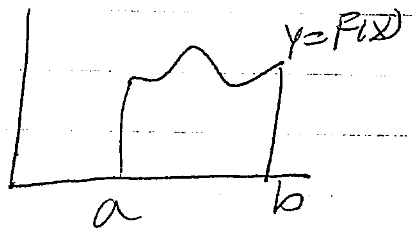
$$\frac{dx}{dt} = 2t \quad \frac{d^2x}{dt^2} = 2 \quad \frac{dy}{dt} = 3t^2 - 7 \quad \frac{d^2y}{dt^2} = 6t$$

$$\frac{d^2y}{dx^2} = \frac{(6t)(2t) - (2)(3t^2 - 7)}{(2t)^3} = \frac{6t^2 + 14}{8t^3} = \frac{3t^2 + 7}{4t^3}$$

So for $t > 0$, $\frac{d^2y}{dx^2} > 0$ concave up

$t < 0$, $\frac{d^2y}{dx^2} < 0$ concave down

Area under parametric curves



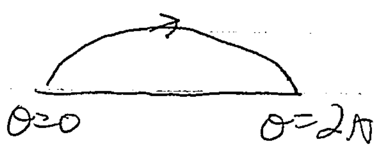
$x = f(t) \quad y = g(t) \quad c \le t \le d,$

$a = f(c) \quad b = f(d).$

Assume this graph is traced just once, i.e., $f(t)$ is strictly increasing on $[c, d]$.

$$\text{Area} = \int_a^b F(x)\,dx = \int_a^b y\,dx = \int_c^d g(t) f'(t)\,dt$$

ex The cycloid



$x = r(\theta - \sin\theta)$
$y = r(1 - \cos\theta)$
$\frac{dx}{d\theta} = r - r\cos\theta$

$$\text{Area} = \int_0^{2\pi} y\,dx = \int_0^{2\pi} r(1-\cos\theta)\cdot r(1-\cos\theta)\,d\theta$$

$$= r^2 \int_0^{2\pi} (1-\cos\theta)^2\,d\theta = r^2 \int_0^{2\pi} 1 - 2\cos\theta + \cos^2\theta\,d\theta$$

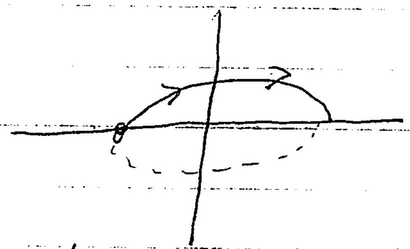
$$= r^2 \int_0^{2\pi} 1 - 2\cos\theta + \tfrac{1}{2}(1+\cos(2\theta))\,d\theta$$

$$= r^2\left[\theta - 2\sin\theta + \tfrac{1}{2}\left(\theta + \tfrac{1}{2}\sin(2\theta)\right)\right]\Big|_0^{2\pi}$$

$$= 3\pi r^2$$

Area of an ellipse

$x = a\sin\theta$
$y = b\cos\theta \qquad -\pi/2 \le \theta \le \pi/2$



$dx = a\cos\theta\, d\theta$

$$\text{Area} = \int_{-\pi/2}^{\pi/2} y\,dx = \int_{-\pi/2}^{\pi/2} b\cos\theta \cdot a \cdot \cos\theta\, d\theta$$

$$= 2ab \int_0^{\pi/2} \cos^2\theta\, d\theta$$
$$= 2ab \int_0^{\pi/2} (1+\cos(2\theta))/2\, d\theta$$
$$= 2ab \left(\theta + \frac{\sin(2\theta)}{2}\right) \cdot \frac{1}{2} \Big|_0^{\pi/2}$$

$$= ab(\pi/2)$$

or $\frac{1}{2}\int_0^{2\pi} y\,dx = \int_0^{\pi} y\,dx = 2\int_0^{\pi/2} y\,dx.$