## Parametric Curve Areas

Area and Direction

Find: $(a)$ $\int_1^2 2x+1dx$ $(b)$ $\int_2^1 2x+1dx$

Solution:

$(a)$ $x^2+x\Big|_1^2 = (4+2)-(1+1)=4$

$(b)$ $x^2+x\Big|_2^1 = (1+1)-(4+2)=-4$

Which one, of the above, represents the area between the curve $y=2x+1$ and the $x$-axis?

"graph"

$(a)$ Since $y>0$ and $dx>0$ the curve is above the $x$-axis and is moving to the right with product>0.

(b) Since $y>0$ and $dx<0$ the curve is above the $x$-axis and is moving to the left with product<0.

Therefore, ($a$) represents the area. Remember, area is always positive!

The same principle applies to parametric curves. Area must be positive.

ie. When we are find the area we require $\int y\,dx = \int y\left(\frac{dx}{dt}dt\right) > 0$!

$\therefore y>0$ and $\frac{dx}{dt}>0$ or $y<0$ and $\frac{dx}{dt}<0$

If $y$ and $\frac{dx}{dt}$ have opposite signs we must change the sign of the integral.

[ Later: For Polar curves we will require $d\theta>0$, to find area since $r^2$ is always positive.]

Example:

Find the area of a simple closed curve defined by parametric equations:

$x=\cos\theta+1 \quad y=\sin\theta+1$

Solution:

"graph"

$y > 0$, for all $\theta$ and

$\dfrac{dx}{d\theta} = -\sin\theta$ and is equal to zero at $\theta = 0, \pi, 2\pi, \ldots$

Hence

| t | 0 | | $\pi$ | | $2\pi$ |
|---|---|---|---|---|---|
| $y$ | | $+$ | | $+$ | |
| $\dfrac{dx}{d\theta}$ | | $-$ | | $+$ | |
| *produce* | | $-$ | | $+$ | |

Therefore,

| $0 \le \theta \le \pi$ | $\pi \le \theta \le 2\pi$ | Result on $[0, 2\pi]$ |
|---|---|---|
| $-A - B - C$ | $A + B$ | $(-A - B - C) + (A + B) = -C$ |

To calculate the area we use,

$$\text{Area} = -\int_0^{2\pi} y \frac{dx}{d\theta} d\theta = -\int_0^{2\pi} (\sin\theta + 1)(-\sin\theta)\, d\theta$$

$$= \int_0^{2\pi} \sin^2\theta + \sin\theta\, d\theta = \int_0^{2\pi} \frac{1 - \cos 2\theta}{2} + \sin\theta d\theta$$

$$= \frac{1}{2}\theta - \frac{\sin 2\theta}{4} - \cos\theta \Big|_0^{2\pi}$$

$$= \frac{1}{2}(2\pi) - 0 - 1 - [0 - 0 - 1] = \pi$$

This represents the area of a circle with radius 1.

Question: Is the area $= 2\int_0^{\pi} (\sin\theta + 1)(\sin\theta)\, d\theta$? [ Look at the diagram!]

Green's Theorem for the area of a simple closed curve:____

Quick parametric sketches in order to find parametric areas

1. Sketch $x$ against $t$
2. Sketch $y$ against $t$
3. Use the above two sketches to sketch $y$ against $x$.

For example: 1.(a) Sketch $x = t^3,\ y = 2t^2 + 1;\ -1 \le t \le 1$

"graphs"

## Areas of Parametric Curves

Depending on the time you have to do this assignment you may wish to set up the integral only.

1. Find the area of the region that lies between the given parametric curve and the $x$-axis.
   (a) $x = t^3,\ y = 2t^2 + 1;\ \ -1 \le t \le 1$
   (b) $x = e^{3t},\ y = e^{-t};\ \ 0 \le t \le \ln 2$
   (c) $x = \cos t,\ y = \sin^2 t;\ \ 0 \le t \le \pi$
   (d) $x = 1 - e^t,\ y = 2t + 1;\ \ 0 \le t \le 1$
   (e) $x = t^2,\ y = t^2 - 2t$ [#8]
   (f) $x = t^2 + 2t - 1,\ \ y = t^2 + t - 2$ [#6]

2. Find the area enclosed by the given curve and the coordinate axes, in the first quadrant.
   (a) $x = 4\cos t,\ y = 5\sin t$
   (b) $x = 4\sin 2t,\ y = 5\sin t$
   (c) $x = \cos t,\ y = e^t - 1$

3. Find the area of each simple closed curve.

(a) $x = t^3 - 3t, \;\; y = 3t^2 - 9$ [#1]

(b) $x = t^2 - 4, \;\; y = t^3 - 4t$ [#2]

(c) $x = t^2 - t, \;\; y = t^3 - 3t$ [#3]

(d) $x = t^3 - 3t^2, \;\; y = t^3 - 3t$ [Example in notes]