Parametric Curves

A curve C consisting of points (x,y) where x and y are both functions of a third variable, say t (called a parameter). Thus, we have each point (x,y) determined by the parametric equations $x = f(t)$ and $y = g(t)$.

Note: Often, but not always, t represents time and these equations are called equations of motion and the curve C is the trajectory of the moving particle.

Examples:

(a) Sketch the curve defined by the parametric equations

$$x = -2 + t$$
$$y = 5 - 2t$$

We will eliminate the parameter t and sketch.

(b) Sketch the curve defined by the parametric equations

$$(x, y) = (1,1) + t(-1,1)$$

This is a compact form of writing the equations separately. Separate this into two individual equations.
Eliminate the parameter and sketch the line.
Find the coordinates of the point of intersection of the above two curves

(c) A spider and a fly crawl so that their positions at time t (in seconds) are:

Spider: $(x, y) = (-2,5) + t(1,-2)$

Fly: $(x, y) = (1,1) + t(-1,1)$

Question: Does the spider meet the fly? Explain

As you should have noticed, parametric equations tend to contain more information.

Eliminating the parameter in parametric equations containing trig functions.

Note: We often use trig identities, especially $\cos^2 t + \sin^2 t = 1$
Examples: Eliminate the parameter t if:

(a) $x = \cos t$, $y = \cos(2t)$

(b) $x = \cos t + 2$, $y = \sin t + 3$

(c) $x = 2\sin t$, $y = 3\cos t$

Slopes of Tangent lines of Parametric Curves

Method:

1. Find $\frac{dy}{dt}$ and $\frac{dx}{dt}$
2. $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$

Example: Find $\frac{dy}{dx}$ if $x = -2 + t \qquad y = 5 - 2t$

Solution:

$$\frac{dy}{dt} = -2 \text{ and } \frac{dx}{dt} = 1. \text{ Hence, } \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-2}{1} = -2$$

Example: Find an equation of the tangent line to the parametric curve

$x = 3 - t^2 \qquad y = 2 + t$ at the point where $t = 1$.

Solution: $\frac{dy}{dt} = 1$ and $\frac{dx}{dt} = -2t$. Hence, $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{1}{-2t}$

Slope of tangent at $t = 1$ is $m_t = \frac{1}{-2} = -\frac{1}{2}$ and $(x,y) = (2, 3)$

Therefore, the equation of the tangent line is: $y - 3 = -\frac{1}{2}(x - 2)$

In order to draw sketches of these parametric curves we also need to be able to find the second derivative to discuss concavity.

Recall from Calculus 1500: To find the second derivative we differentiated the first derivative. That is exactly what we do with parametric equations. Just be careful!

Example: Find the first and second derivatives of the parametric curve

$$x = 3 - t^2 \qquad y = 2 + t$$

Solution: As above, the first derivative is

$$\frac{dy}{dt} = 1 \text{ and } \frac{dx}{dt} = -2t. \text{ Hence, } \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{1}{-2t}$$

That is, $y' = -\frac{1}{2t}$ and the second derivative is

$\frac{d(y')}{dx}$ which most people write as $\frac{d^2y}{dx^2}$. In other words, you find the

parametric derivative using $x = 3 - t^2 \qquad NewY = y' = -\frac{1}{2t} = -\frac{1}{2}t^{-1}$

So $\frac{d^2y}{dx^2} = \frac{d(NewY)}{dx} = \frac{\frac{1}{2}t^{-2}}{-2t} = \frac{1}{-4t^3}$

Be very careful not to confuse this with the derivative of a quotient. It has nothing to do with quotients!

Example: Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at $t = 2$ if $x = \frac{1}{2}t^2 \quad y = \frac{1}{3}t^3$

Solution: $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{t^2}{t} = t$ and $\frac{d^2y}{dx^2} = \frac{d\left(\frac{dy}{dx}\right)}{dx} = \frac{1}{t}$

When $t = 2 \qquad \frac{dy}{dx} = 2$ and $\frac{d^2y}{dx^2} = \frac{1}{2}$

We are now in a position to follow curve sketching steps, very similar to the curve sketching steps you followed in math 1500.

Other than plotting some interesting points we will follow these steps:

1. Find the x and y intercepts.
2. Determine the "starting" and "ending" locations of the curve by finding the limits of x and y as $t \to \pm\infty$.
3. Find all horizontal and vertical tangent lines.
4. Use a first derivative chart to find the direction the curve is following in strategic intervals.
5. Sketch the curve. Using the above information.
6. Check the concavity of your curve using the values of the second derivative in strategic intervals.

Example: Follow the above steps in order to sketch the curve determined by the parametric equations $x = t^3 - 3t^2 \qquad y = t^3 - 3t$

Solution:

<u>Intercepts</u>

$x = 0$	$y = 0$
$0 = t^3 - 3t^2$	$0 = t^3 - 3t$
$0 = t^2(t-3)$	$0 = t\left(t-\sqrt{3}\right)\left(t+\sqrt{3}\right)$
$t = 0, 3$	$t = 0, \sqrt{3}, -\sqrt{3}$
$\therefore y-$intercepts are: $0, 18$	$x-$intercepts are: $0, 3\sqrt{3}-9, -3\sqrt{3}-9$

<u>Beginning and Ending Positions</u>

$\lim\limits_{t\to-\infty} x = \lim\limits_{t\to-\infty}\left(t^3-3t^2\right) = \lim\limits_{t\to-\infty}\left[t^2(t-3)\right] = -\infty$ ($\therefore$ starts far to the left)

$\lim\limits_{t\to\infty} x = \lim\limits_{t\to\infty}\left(t^3-3t^2\right) = \lim\limits_{t\to\infty}\left[t^2(t-3)\right] = \infty$ ($\therefore$ ends far to the right)

$\lim\limits_{t\to\pm\infty} y = \lim\limits_{t\to\pm\infty}\left(t^3-3t\right) = \lim\limits_{t\to\pm\infty}\left[t\left(t^2-3\right)\right] = \pm\infty$

($\therefore$ starts far down and ends far up)

<u>Coordinates of points of horizontal and vertical tangent lines.</u> (These are our horizontal and vertical "min/max" positions.)

Be very careful – we will be using $\dfrac{dy}{dt}$ and $\dfrac{dx}{dt}$ NOT $\dfrac{dy}{dx}$!!!

$\frac{dy}{dt} = 3t^2 - 3$

$0 = 3(t-1)(t+1)$

$t = 1, -1$ and $\frac{dx}{dt} \neq 0$

$\therefore$ The curve has horizontal tangents at $(-2,-2)$ and $(-4,2)$

$\frac{dx}{dt} = 3t^2 - 6t$

$0 = 3t(t-2)$

$t = 0, 2$ and $\frac{dy}{dt} \neq 0$

$\therefore$ The curve has vertical tangents at $(0,0)$ and $(-4,2)$

<u>The Chart</u>

We will use the critical values of t to determine the directions of the curve

t		-1		0		1		2	
$\frac{dx}{dt}$	$+$		$+$		$-$		$-$		$+$
$\frac{dy}{dt}$	$+$		$-$		$-$		$+$		$+$
x	$\rightarrow$		$\rightarrow$		$\leftarrow$		$\leftarrow$		$\rightarrow$
y	$\uparrow$		$\downarrow$		$\downarrow$		$\uparrow$		$\uparrow$
curve	$\nearrow$		$\searrow$		$\swarrow$		$\nwarrow$		$\nearrow$

Use the above information, especially the curve directions in the chart to sketch the curve.

<u>Check the concavity</u> [Note: Now we will be using $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.]

$$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{3t^2-3}{3t^2-6t}\quad\left[\begin{array}{c}\text{Note: It is tempting but not advisable}\\ \text{to reduce this fraction.}\end{array}\right]$$

$$\frac{d^2y}{dx^2}=\frac{\frac{d\left(\frac{3t^2-3}{3t^2-6t}\right)}{dt}}{\frac{dx}{dt}}=\frac{\frac{6t\left(3t^2-6t\right)-\left(6t-6\right)\left(3t^2-3\right)}{\left(3t^2-6t\right)^2}\leftarrow\text{[This is the Quotient Rule]}}{3t^2-6t}$$

$$=\frac{18t^3-36t^2-18t^3+18t^2+18t-18}{\left(3t^2-6t\right)^3}$$

$$=\frac{-18t^2+18t-18}{\left(3t^2-6t\right)^3}=\frac{-18\left(t^2-t+1\right)}{27t^3\left(t-2\right)^3}$$

Therefore, the numbers which appear on the number line are t = 0 and 2.

t		0		2	
$\frac{d^2y}{dx^2}$	$-$		$+$		$-$
curve	$\cap$		$\cup$		$\cap$

Practice: Follow the steps outlined above to analyze and sketch the curve described by the parametric equations:

1. $x=t^3-3t$ $\quad y=3t^2-9$
2. $x=t^2-4$ $\quad y=t^3-4t$
3. $x=t^2-t$ $\quad y=t^3-3t$
4. $x=t^2+2t$ $\quad y=t^2+t$
5. $x=t^2$ $\quad y=t^2-2t$
6. $x=t^2+2t-1$ $\quad y=t^2+t-2$
7. $x=t^3-3t$ $\quad y=t^3-3t^2$
8. $x=t^3$ $\quad y=t^2-2t$
9. $x=t^3-3t$ $\quad y=t^2-2t$
10. $x=e^t-t$ $\quad y=4e^{\frac{t}{2}}$