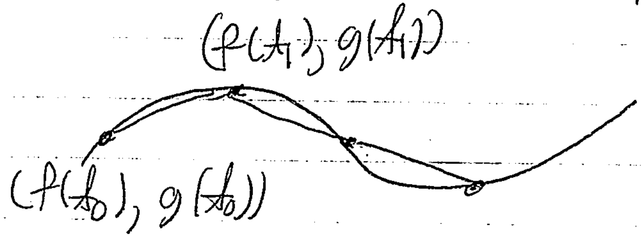
Arc Length for parametric



$$\text{approximate length} = \sum_{i=1}^{n} \sqrt{\left(f(t_i) - f(t_{i-1})\right)^2 + \left(g(t_i) - g(t_{i-1})\right)^2}$$

By the Mean Value Theorem

$$\frac{f(t_i) - f(t_{i-1})}{t_i - t_{i-1}} = f'(t_i^*)$$

$$\therefore \text{ approximate length} = \sum_{i=1}^{n} \left(f'(t_i^*)^2 + g'(t_i^{**})^2\right)^{1/2} \Delta t$$

$$= \lim_{n \to \infty} \quad ''$$

$$= \int_a^b \left(f'(t)^2 + g'(t)^2\right)^{1/2} dt$$

$dx = f'(t)\, dt \qquad dy = g'(t)\, dt$

$(dx)^2 + (dy)^2 = \left(f'(t)^2 + g'(t)^2\right) dt$

$(ds)^2 = (dx)^2 + (dy)^2 \qquad \text{arc length} = \int_a^b ds$

arc length of a cycloid.

$x = r(\theta - \sin\theta)$ $\quad \frac{dx}{d\theta} = r - \cos\theta\, r \quad \frac{dy}{d\theta} = \sin\theta\, r$

$y = r(1-\cos\theta)$

$$\left(\frac{dy}{d\theta}\right)^2 + \left(\frac{dx}{d\theta}\right)^2 = (r(1-\cos\theta))^2 + r^2\sin^2\theta$$

$$= r^2 - 2r^2\cos\theta + r^2\cos^2\theta + r^2\sin^2\theta$$

$$= 2r^2(1-\cos\theta)$$

$$= 2r^2(2\sin(\theta/2)^2)$$

$$= (2r\sin(\theta/2))^2$$

$$\int_0^{2\pi} ((2r\sin(\theta/2))^2)^{1/2}\, d\theta$$

$$= \int_0^{2\pi} 2r\sin(\theta/2)\, d\theta$$

$$= -2r\cos(\theta/2)(2)\Big|_0^{2\pi}$$

$$= -2r(-1-1)(2) = 8r$$

Surface Area

$x = f(t)$ $\quad a \le t \le b$
$y = g(t)$

$$\text{Surface Area} = \int_a^b 2\pi y \, ds$$
$$= \int_a^b 2\pi g(t) \left(f'(t)^2 + g'(t)^2 \right)^{1/2} dt$$

Ex $\quad \cancel{x} = r\cos\theta \quad -\pi/2 \le \theta \le \pi/2$
$\quad x = r\sin\theta$

$\frac{dx}{d\theta} = r\cos\theta \quad \frac{dy}{d\theta} = -r\sin\theta$

$$SA = 2\pi \int_{-\pi/2}^{\pi/2} r\cos\theta \left((-r\sin\theta)^2 + (r\cos\theta)^2 \right)^{1/2} d\theta$$
$$= 2\pi \int_{-\pi/2}^{\pi/2} r\cos\theta \left(r^2\sin^2\theta + r^2\cos^2\theta \right)^{1/2} d\theta$$
$$= 2\pi \int_{-\pi/2}^{\pi/2} r\cos\theta \left(r^2 \right)^{1/2} d\theta$$
$$= 2\pi r^2 \int_{-\pi/2}^{\pi/2} \cos\theta \, d\theta = 2\pi r^2 \sin\theta \Big|_{-\pi/2}^{\pi/2} = 4\pi r^2$$

or $\quad 2\pi \cdot 2 \cdot \int_0^{\pi/2} r^2 \cos\theta \, d\theta$