Partial Fractions

Do one of the following two problems. Preferably the easier one.

Find: (a) $\int \frac{6x}{x^2+x-2}dx$ (b) $\int\left(\frac{2}{x-1}+\frac{4}{x+2}\right)dx$

Express as a single fraction: $\frac{1}{x-1}+\frac{2}{x+1}=$

Express as a sum of partial fractions: $\frac{4}{(x-2)(x+2)}$. This will be a major part of this section.

Identities

Find the values of the constants A, B and C such that

$$5x+3=Ax(x+3)+Bx(x-1)+C(x-1)(x+3)$$

Method 1. Comparing coefficients.

$$5x+3=(A+B+C)x^2+(3A-B+2C)x-3C$$

$$\therefore 0=A+B+C \qquad 5=3A-B+2C \qquad 3=-3C$$

$$Hence,\ C=-1 \qquad and \qquad 0=A+B-1$$

$$5=3A-B+2(-1)$$

$$\left.\begin{aligned}\therefore A+B&=1\\ 3A-B&=7\end{aligned}\right\}\Rightarrow 4A=8\Rightarrow A=2\ and\ B=-1$$

Method 2 identities

$$Let\ x=-3\ then\ -15+3=B(-3)(-4) \qquad or\ B=-1$$

$$Let\quad x=1\quad then\quad 5+3=A(1)(4)\quad and\quad A=2$$

To find the value of C we could compare the constants and arrive at

$$3=-3C\quad \therefore C=-1$$

You may use either method or a combination of both.

We will look at three types of partial fractions:

1. Denominator contains only linear factors.

$$\frac{11x+12}{(2x+3)(x^2-x-6)}=\frac{11x+12}{(2x+3)(x-3)(x+2)}=\frac{A}{2x+3}+\frac{B}{x-3}+\frac{C}{x+2}$$

2. Denominator contains quadratic(non-factorable) factors.

$$\frac{3x+1}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1}$$

3. Denominator contains repeated factors

$$\frac{1}{(x+2)(x^2-2x+1)} = \frac{1}{(x+2)(x-1)^2} = \frac{A}{x+2} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$$

Once you've performed the partial decomposition you are ready to evaluate the integral.
Let's return to our original problem.

Example 1: Find $\int \frac{6x}{x^2+x-2}dx$

Solution:

$$\int \frac{6x}{x^2+x-2}dx = \int \frac{6x}{(x+2)(x-1)}dx = \int \frac{A}{x+2} + \frac{B}{x-1}dx$$

$$\frac{6x}{(x+2)(x-1)} = \frac{A}{x+2} + \frac{B}{x-1}$$

$$6x = A(x-1) + B(x+2)$$

$$\begin{array}{lll} & Let\ \ x=1 & Let\ \ x=-2 \\ & 6=3B & -12=-3A \\ hence & B=2 & A=4 \end{array}$$

$$\therefore \int \frac{6x}{(x+2)(x-1)}dx = \int \frac{4}{x+2} + \frac{2}{x-1}dx$$

$$= 4\ln|x+2| + 2\ln|x-1| + C$$

Example 2: Find $\int \frac{7x+2}{x^2(3x+1)}dx$

Solution:

$$\frac{7x+2}{x^2(3x+1)}=\frac{A}{x}+\frac{B}{x^2}+\frac{C}{3x+1}$$

$$7x+2=Ax(3x+1)+B(3x+1)+Cx^2$$

$Let\ \ x=0$	$Let\ \ x=-\frac{1}{3}$	<u>compare x^2coefficients</u>
$2=B$	$-\frac{7}{3}+2=\frac{1}{9}C$	$0=3A+C$
	$-\frac{1}{3}=\frac{1}{9}C$	$0=3A-3$
	$-3=C$	$A=1$

$$\therefore\int\frac{7x+2}{x^2(3x+1)}dx=\int\frac{1}{x}+\frac{2}{x^2}-\frac{3}{3x+1}dx$$

$$=\ln|x|-\frac{2}{x}-\ln|3x+1|+C$$

Example 3: Find $\int\frac{2x^2+2x+3}{(x+2)(x^2+3)}dx$

Solution:

$$\int \frac{2x^2+2x+3}{(x+2)(x^2+3)}dx = \int \frac{A}{x+2} + \frac{Bx+C}{x^2+3}dx$$

$$2x^2+2x+3 = A(x^2+3)+(Bx+C)(x+2)$$

$Let\ \ x=-2$	<u>compare x^2 coef.</u>	<u>compare constants</u>
$8-4+3=7A$	$2=A+B$	$3=3A+2C$
$7=7A$	$2=1+B$	$3=3+2C$
$A=1$	$B=1$	$C=0$

$$\therefore \int \frac{2x^2+2x+3}{(x+2)(x^2+3)}dx = \int \frac{1}{x+2} + \frac{x}{x^2+3}dx$$

$$= \ln|x+2| + \frac{1}{2}\int \frac{2x}{x^2+3}dx = \ln|x+2| + \frac{1}{2}\ln|x^2+3| + C$$

Example 3: Find $\int \frac{x^3-18x-21}{(x+2)(x-5)}dx$

Solution:

Since the degree of the numerator is ≥the degree of the denominator, we have changed this improper fraction in a "whole number" and a proper fraction.

$$\int \frac{x^3-18x-21}{(x+2)(x-5)}dx=\int \frac{x^3-18x-21}{x^2-3x-10}dx=\int\left(x+3+\frac{x+9}{x^2-3x-10}\right)dx$$

$$But\ \frac{x+9}{x^2-3x-10}=\frac{A}{x+2}+\frac{B}{x-5}$$

$$x+9=A(x-5)+B(x+2)$$

$$\begin{array}{ll} Let\ \ x=5 & Let\ \ x=-2 \\ 14=7B & 7=-7A \\ B=2 & A=-1 \end{array}$$

$$\therefore \int \frac{x^3-18x-21}{(x+2)(x-5)}dx=\int\left(x+3-\frac{1}{x+2}+\frac{2}{x-5}\right)dx$$

$$=\frac{x^2}{2}+3x-\ln|x+2|+2\ln|x-5|+C$$

This completes our techniques of integration. This does not mean that we have exhausted <u>all</u> the techniques. We have exhausted all the techniques you will encounter in this course.

* Stewart has an excellent section on strategies for integration pg.505 section 7.5. It is recommended, very strongly, that you read this section and try as many problems as you can. Some unusual problems are numbers: 31, 35, 39, 49, and 53 to end.

Many integrals can be solved in more than one way. Try the following for some fun. However, <u>you</u> should always attempt the most efficient method.

1. Evaluate $\int \frac{x^3}{\sqrt{4+x^2}}dx$ Using:

   (a) trig substitution (b) substitution letting $u=\sqrt{4+x^2}$

   (c) by parts with $dv=\left(\frac{x}{\sqrt{4+x^2}}\right)dx$

2. Evaluate $\int x\sqrt{4+x}\,dx$ Using:

   (a) trig substitution (b) substitution letting $u=\sqrt{4+x}$

(c) substitution letting $u = 4 + x$ (c) by parts with $dv = \sqrt{4+x}dx$

Solutions:

1. (a) $\quad Let\ x = 2\tan\theta \qquad x^2 = 4\tan^2\theta$

$$\int \frac{x^3}{\sqrt{4+x^2}}dx = \int \frac{8\tan^3\theta\left(2\sec^2\theta d\theta\right)}{\sqrt{4+4\tan^2\theta}} = 8\int \tan^3\theta\sec\theta\, d\theta \qquad dx = 2\sec^2\theta d\theta$$

$$= 8\int\left(\sec^2\theta - 1\right)\left(\sec\theta\tan\theta\right)d\theta = 8\left[\frac{\sec^3\theta}{3} - \sec\theta\right] + C$$

$$= 8\left[\frac{\frac{\left(\sqrt{x^2+4}\right)^3}{8}}{3} - \frac{\sqrt{x^2+4}}{2}\right] + C \qquad "sketch"$$

(b) $\quad Let\quad u = \sqrt{4+x^2} \qquad then \quad u^2 = 4 + x^2 \qquad and \quad 2udu = 2xdx$

$$\int \frac{x^3}{\sqrt{4+x^2}}dx = \int \frac{x^2\left(xdx\right)}{\sqrt{4+x^2}} = \int \frac{\left(u^2-4\right)udu}{u}$$

$$= \frac{u^3}{3} - 4u + C = \frac{\left(\sqrt{4+x^2}\right)^3}{3} - 4\sqrt{4+x^2} + C$$

(c) $\quad Let\quad u = x^2 \qquad dv = \frac{xdx}{\sqrt{4+x^2}}$

$$du = 2xdx \qquad v = \left(4+x^2\right)^{\frac{1}{2}}$$

$$\int \frac{x^3}{\sqrt{4+x^2}}dx = x^2\left(4+x^2\right)^{\frac{1}{2}} - \int\left(4+x^2\right)^{\frac{1}{2}}\left(2xdx\right)$$

$$= x^2\left(4+x^2\right)^{\frac{1}{2}} - \frac{2}{3}\left(4+x^2\right)^{\frac{3}{2}} + C$$

Which method was the easiest?

2.(a) $\int x\sqrt{4+x}\,dx$ $\qquad$ *Let* $x = 4\tan^2\theta$

$= \int 4\tan^2\theta\sqrt{4+4\tan^2\theta}\left(8\tan\theta\sec^2\theta d\theta\right)$ $\qquad$ $dx = 8\tan\theta\sec^2\theta d\theta$

$= 64\int \tan^2\theta\sqrt{\sec^2\theta}\left(\tan\theta\sec\theta d\theta\right)$

$= 64\int\left(\sec^2\theta - 1\right)\sec^2\theta\left(\tan\theta\sec\theta d\theta\right) = 64\int\left(\sec^4\theta - \sec^2\theta\right)\left(\tan\theta\sec\theta d\theta\right)$

$= 64\left[\dfrac{\sec^5\theta}{5} - \dfrac{\sec^3\theta}{3}\right] + C = 64\left[\dfrac{(x+4)^2\sqrt{x+4}}{5(32)\sqrt{\ }} - \dfrac{(x+4)\sqrt{x+4}}{3(8)\sqrt{\ }}\right] + C$

(b) $\qquad$ *Let* $u = \sqrt{x+4}$ $\qquad$ *then* $u^2 = x+4$ *and* $2udu = dx$

$\int x\sqrt{4+x}\,dx = \int\left(u^2-4\right)u\left(2udu\right) = \int\left(2u^4 - 8u^2\right)du$

$= \dfrac{2u^5}{5} - \dfrac{8u^3}{3} + C = \dfrac{2\left(\sqrt{x+4}\right)^5}{5} - \dfrac{8\left(\sqrt{x+4}\right)^3}{3} + C$

(c) $\qquad$ *Let* $u = x+4$ $\qquad$ *then* $du = dx$

$\int x\sqrt{4+x}\,dx = \int (u-4)\sqrt{u}\,(du) = \int\left(u^{\frac{3}{2}} - 4u^{\frac{1}{2}}\right)du$

$= \dfrac{2u^{\frac{5}{2}}}{5} - \dfrac{8u^{\frac{3}{2}}}{3} + C = \dfrac{2(x+4)^{\frac{5}{2}}}{5} - \dfrac{8(x+4)^{\frac{3}{2}}}{3} + C$

(d) *Let* $u = x$ $\qquad$ $dv = (4+x)^{\frac{1}{2}}\,dx$

$du = dx$ $\qquad$ $v = \dfrac{2}{3}\left(4+x^2\right)^{\frac{3}{2}}$

$\int x\sqrt{4+x}\,dx = \dfrac{2}{3}x(4+x)\sqrt{4+x} - \dfrac{2}{3}\int\left(4+x^2\right)^{\frac{3}{2}}\,dx$

$= \dfrac{2}{3}x(4+x)\sqrt{4+x} - \dfrac{4}{15}\left(4+x^2\right)^{\frac{5}{2}} + C$

Which method was the easiest?

Logarithmic pattern: $\int \frac{f'(x)}{f(x)} = \ln|f(x)| + C$

Find: $(a)\ \int \frac{x}{x^2+5}dx \qquad (b)\ \int \frac{e^{2x}}{1-e^{2x}}dx \qquad (c)\ \int \frac{\sec^2 x+1}{\tan x + x}dx \qquad (d)\ \int \frac{1}{x\ln x}dx$

Solutions:

$$(a)\qquad \int \frac{x}{x^2+5}dx = \frac{1}{2}\int \frac{2x}{x^2+5}dx = \frac{1}{2}\ln|x^2+5| + C$$

$$(b)\qquad \int \frac{e^{2x}}{1-e^{2x}}dx = -\frac{1}{2}\int \frac{-2e^{2x}}{1-e^{2x}}dx = -\frac{1}{2}\ln|1-e^{2x}| + C$$

$$(c)\qquad \int \frac{\sec^2 x+1}{\tan x + x}dx = \ln|\tan x + x| + C$$

$$(d)\qquad \int \frac{1}{x\ln x}dx = \int \frac{\frac{1}{x}}{\ln x}dx = \ln|\ln x| + C$$

Square root pattern :

Let u = root and use substitution.

Find: $(\mathrm{a})\ \int \frac{\cos\sqrt{x}}{\sqrt{x}}dx \qquad (b)\ \int \frac{e^x}{\sqrt{e^x+5}}dx \qquad (c)\ \int \sin\sqrt{x}\,dx \qquad (d)\ \int e^{\sqrt{x}}dx$

Solutions:

$$(a)\qquad Let\ u = \sqrt{x} \qquad then \qquad du = \frac{1}{2\sqrt{x}}dx$$

$$\int \frac{\cos\sqrt{x}}{\sqrt{x}}dx = \int 2\cos u\,du = 2\sin u + C = 2\sin\sqrt{x} + C$$

$$(b)\qquad Let\ u = \sqrt{e^x+5} \qquad u^2 = e^x + 5 \qquad 2udu = e^x dx$$

$$\int \frac{e^x}{\sqrt{e^x+5}}dx = \int \frac{2u\,du}{u} = 2\int du = 2u + C = 2\sqrt{e^x+5} + C$$

(c) $\quad Let\ u=\sqrt{x} \qquad u^2=x \qquad 2udu=dx$

$$I=\int \sin\sqrt{x}\,dx=\int \sin u\,(2udu)$$ Using integration by parts

$Let\ w=u \qquad dv=\sin u du$

$dw=du \qquad v=-\cos u$

$$I=-2u\cos u+2\int \cos u du=-2u\cos u+2\sin u+C=-2\sqrt{x}\cos\sqrt{x}+2\sin\sqrt{x}+C$$

(d) $\quad Let\ u=\sqrt{x} \qquad u^2=x \qquad 2udu=dx$

$$I=\int e^{\sqrt{x}}\,dx=\int e^{u}\,(2udu)$$ Using integration by parts

$Let\ w=u \qquad dv=e^{u}du$

$dw=du \qquad v=e^{u}$

$$I=2ue^{u}-2\int e^{u}du=2ue^{u}-2e^{u}+C=2\sqrt{x}e^{\sqrt{x}}-2e^{\sqrt{x}}+C$$