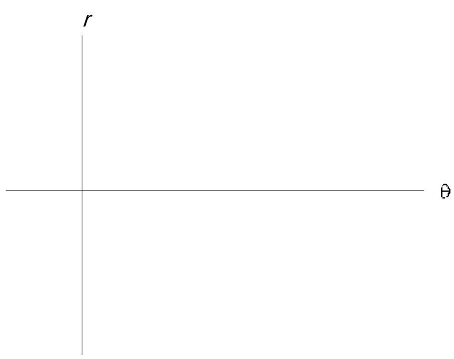
Polar Curves

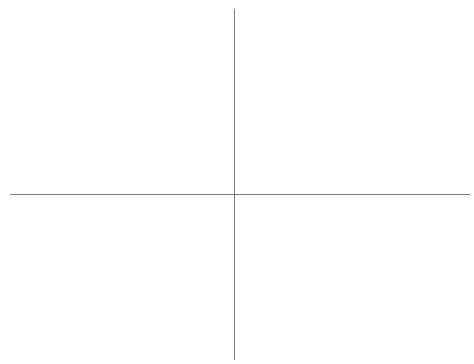
We will use a two step process to draw polar graphs.
Step 1. Sketch *r* against the parameter, usually θ.
Step 2. Use the sketch from step 1 to draw the polar sketch.

Example: Sketch $r = 3 + 3\sin\theta$

r verses θ

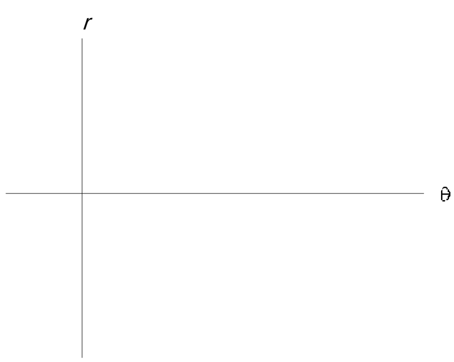


polar graph

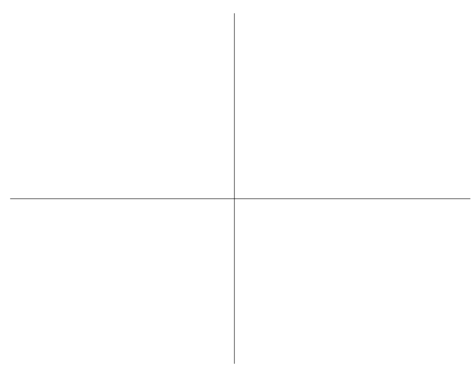


Example: Sketch $r = 3 + 5\sin\theta$

r verses θ

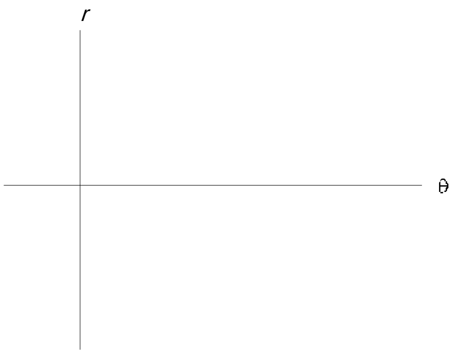


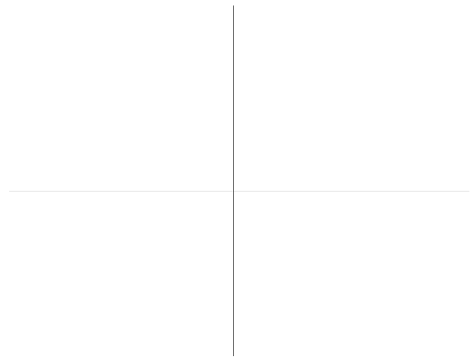
polar graph



Example: Sketch $r = 3 + 2\sin\theta$

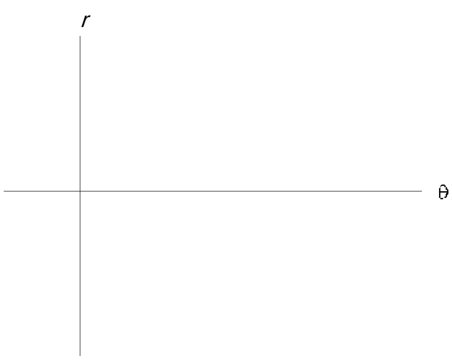
r verses θ polar graph

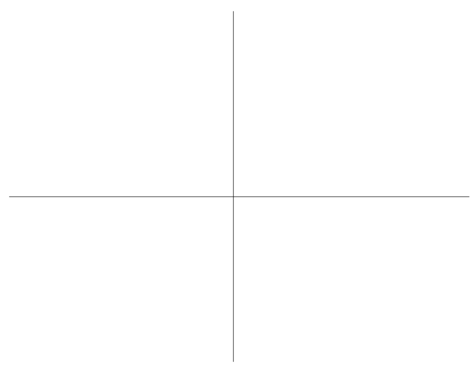




Example: Sketch $r = 5 + 2\sin\theta$

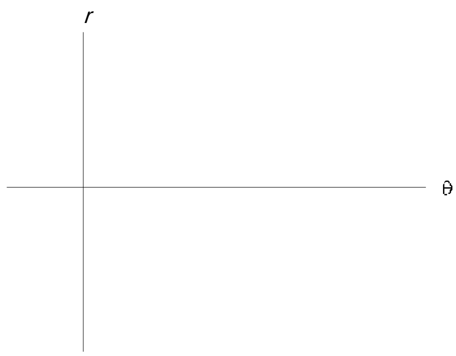
r verses θ polar graph

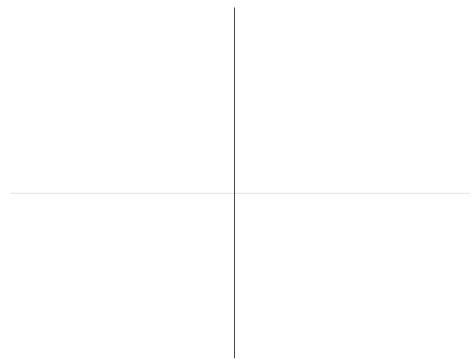




Example: Sketch $r = 3\sin\theta$

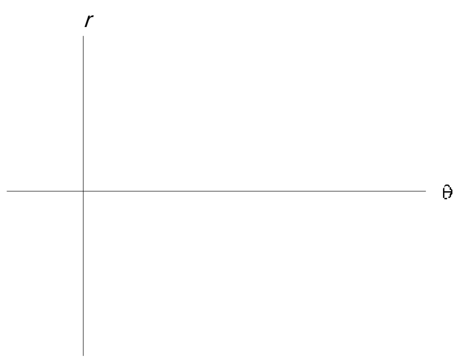
$\underline{r \text{ verses } \theta}$ polar graph

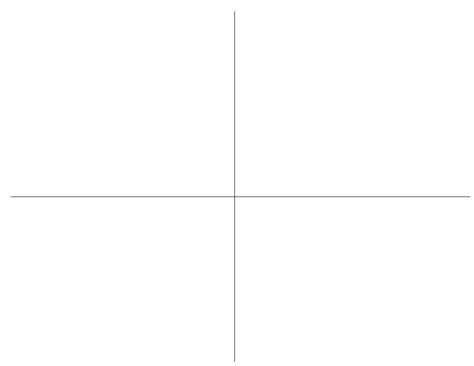




Example: Sketch $r = 3\sin 2\theta$

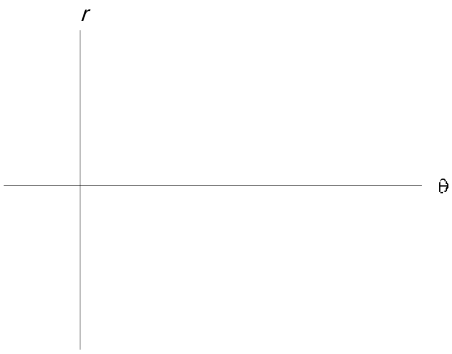
$\underline{r \text{ verses } \theta}$ polar graph



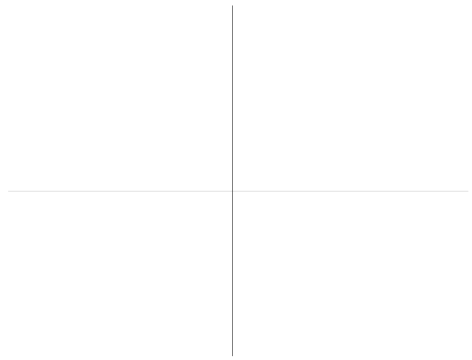


Example: Sketch $r = 3\sin 3\theta$

<u>r verses θ</u>

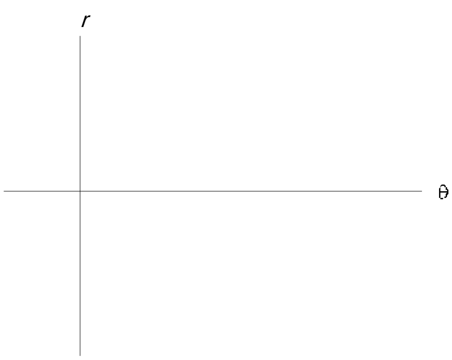


<u>polar graph</u>

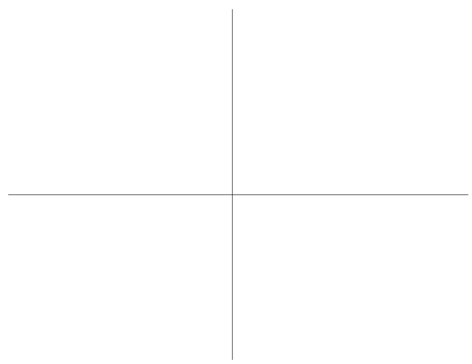


Example: Sketch $r = -3\sin\theta$

<u>r verses θ</u>

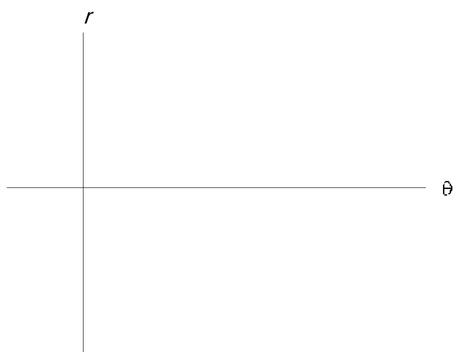


<u>polar graph</u>

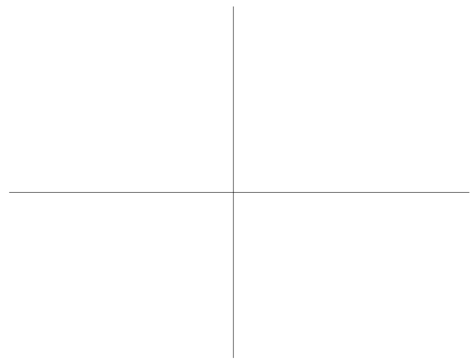


Example: Sketch $r = \theta,\ \ \theta \geq 0$

<u>r verses θ</u>

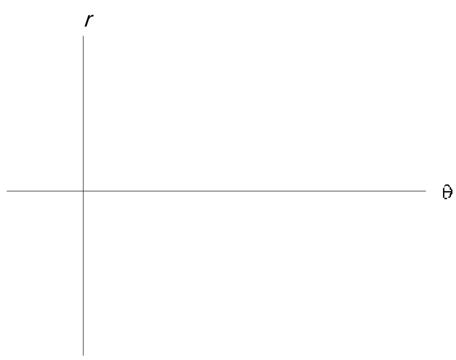


<u>polar graph</u>

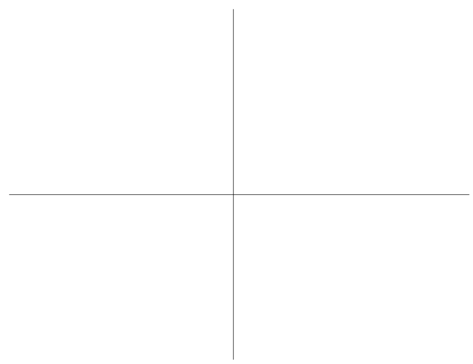


Example: Sketch $r = \theta^2 - \pi\theta, \qquad 0 \leq \theta \leq 2\pi$

<u>r verses θ</u>



<u>polar graph</u>



This may be a good time for you to try a set of grouped polar graphs so that you can come up with some useful generalizations. Try this exercise for homework.
Sketch:

Lines:

1. $r\cos\theta = 3$ 2. $r\sin\theta = 2$ 3. $\theta = \dfrac{3\pi}{4}$ 4. $r = 4\sec\theta$ 5. $2r\cos\theta + 3r\sin\theta + 6 = 0$

Circles:

1. $r = 3$ 2. $r = 4\cos\theta$ 3. $r = 5\sin\theta$ 4. $r = 10\sin\theta$ 5. $r = 2$

6. $r = -5\sin\theta$ 7. $r = -8\cos\theta$ 8. $r = a\sin\theta$ 9. $r = a\cos\theta$ $a > 0$

Cardioids and limacons: $\{r = a \pm b\sin\theta \ \ and \ \ r = a \pm b\cos\theta\}$

1. $r = 5 + 2\sin\theta$ 2. $r = 5 - 2\sin\theta$ 3. $r = 5 + 2\cos\theta$ 4. $r = 5 - 2\cos\theta$

$\boxed{a > 2b}$ ← convex limacon

5. $r = 3 + 2\sin\theta$ 6. $r = 3 - 2\sin\theta$ 7. $r = 3 + 2\cos\theta$ 8. $r = 3 - 2\cos\theta$

$\boxed{b < a < 2b}$ ← dimpled limacon

9. $r = 2 + 2\sin\theta$ 10. $r = 2 - 2\sin\theta$ 11. $r = 2 + 2\cos\theta$ 12. $r = 2 - 2\cos\theta$

$\boxed{a = b}$ ← cardioid

13. $r = 2 + 5\sin\theta$ 14. $r = 2 - 5\sin\theta$ 15. $r = 2 + 5\cos\theta$ 16. $r = 2 - 5\cos\theta$

$\boxed{a < b}$ ← limacon with inner loop

Note: If $a = 0$ or $b = 0$ 17. $r = 2$ 18. $r = 2\sin\theta$ ← circles

Rose Curves:

$\{r = a\sin n\theta \ \ and \ \ r = a\cos n\theta\}$ ← differentiate between odd and even n

19. $r = 3\cos\theta$ 20. $r = 3\cos 3\theta$ 21. $r = -3\cos\theta$ 22. $r = -3\cos 3\theta$

23. $r = 3\sin\theta$ 24. $r = 3\sin 3\theta$ 25. $r = -3\sin\theta$ 26. $r = -3\sin 3\theta$

27. $r = 3\cos 2\theta$ 28. $r = 3\cos 4\theta$ 29. $r = -3\cos 2\theta$ 30. $r = 3\sin 2\theta$

31. $r = 3\sin 4\theta$ 32. $r = -3\sin 2\theta$

Lemniscates: $\left\{r^2 = \pm a^2 \cos 2\theta \;\; and \;\; r^2 = \pm a^2 \sin 2\theta\right\}$

33. $r^2 = 4\cos 2\theta$ 34. $r^2 = -4\cos 2\theta$ 35. $r^2 = 4\sin 2\theta$ 36. $r^2 = -4\sin 2\theta$

Spirals: $\left\{r = a\theta \text{ where } \theta \le 0 \, or \, \theta \ge 0\right\}$

37. $r = \theta,\ \theta \ge 0$ 38. $r = 2\theta,\ \theta \ge 0$ 39. $r = \theta,\ \theta \le 0$ 40. $r = 2\theta,\ \theta \le 0$

Tangents to Polar Curves

Two steps:

1. Change to *x* and *y*.
2. Find the derivative as a pair of parametric equations
3. Substitute the θ value(s) to find the slope and the coordinates of the point. Be sure that the denominator $\frac{dx}{d\theta} \ne 0$. If the denominator has a value of 0, the line may be vertical or we may have to use L'Hopital's Rule.

Example: Find an equation of the tangent line to the curve

$$r = 1 + \sin\theta \text{ when } \theta = \frac{\pi}{4}$$

Solution:

$$x = r\cos\theta = (1+\sin\theta)\cos\theta \quad \text{and} \quad y = r\sin\theta = (1+\sin\theta)\sin\theta$$

$$\therefore \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\cos\theta \cdot \sin\theta + (1+\sin\theta)(\cos\theta)}{\cos\theta \cdot \cos\theta + (1+\sin\theta)(-\sin\theta)}$$

OR

$$\frac{dy}{dx} = \frac{r'\sin\theta + r\cos\theta}{r'\cos\theta - r\sin\theta}$$

When $\theta = \dfrac{\pi}{4}$ then

$$x = \left(1 + \frac{\sqrt{2}}{2}\right)\frac{\sqrt{2}}{2} = \frac{\sqrt{2}+1}{2} \quad \text{and} \quad y = \left(1 + \frac{\sqrt{2}}{2}\right)\frac{\sqrt{2}}{2} = \frac{\sqrt{2}+1}{2}$$

$$m_t = \frac{\dfrac{\sqrt{2}}{2}\cdot\dfrac{\sqrt{2}}{2} + \left(1+\dfrac{\sqrt{2}}{2}\right)\dfrac{\sqrt{2}}{2}}{\dfrac{\sqrt{2}}{2}\cdot\dfrac{\sqrt{2}}{2} - \left(1+\dfrac{\sqrt{2}}{2}\right)\dfrac{\sqrt{2}}{2}} = \frac{\dfrac{1}{2} + \dfrac{\sqrt{2}+1}{2}}{\dfrac{1}{2} - \dfrac{\sqrt{2}+1}{2}} = \frac{\dfrac{2+\sqrt{2}}{2}}{-\dfrac{\sqrt{2}}{2}} = -\frac{2+\sqrt{2}}{\sqrt{2}} = -\left(\sqrt{2}+1\right)$$

Therefore, the equation of the line is: $y - \left(\dfrac{\sqrt{2}+1}{2}\right) = -\left(\sqrt{2}+1\right)\left(x - \left[\dfrac{\sqrt{2}+1}{2}\right]\right)$.

Generally, this type of question takes a lot of work!

<u>Tangents at the pole</u>

These are the easiest to find and especially to write, since any line through the pole has the equation $r = \theta$ where θ is the angle the line makes with the polar axis. [Note: At the pole $r = 0$, therefore check if $r' = 0$ to see if the derivative may require L'Hopital's Rule.]

Example: Find tangents to the curve $r = \cos(2\theta)$ at the pole.

Solution: At the pole $r = 0$

Hence, $r = 0$

$$0 = \cos(2\theta)$$

$$2\theta = \frac{\pi}{2}, \frac{3\pi}{2}$$

$$\theta = \frac{\pi}{4}, \frac{3\pi}{4} \text{ on } 0 \le \theta \le 2\pi$$

These are the angles the tangent lines make with the polar axis.

Hence, equations are $\theta = \frac{\pi}{4},\ \theta = \frac{3\pi}{4}\left(\begin{array}{l}\text{provided } r' \neq 0, \text{ which} \\ \text{makes the denominator } \frac{dx}{d\theta} \neq 0\end{array}\right)$

“**graph**”

If $r' = 0$ we would have to investigate. If $\frac{dx}{d\theta}$ and $\frac{dy}{d\theta}$ are both equal to zero we can take the limits as θ approach the angle to determine whether this limit exists. We may use L’Hopital’s Rule to determine this limit.

Example: Does the curve $r = 1 - \cos\theta$ have a tangent line at the pole(origin)?

Solution:

If $r=0$ then $0=1-\cos\theta$ or $\cos\theta=1$

$\therefore \theta=0$ on $0\le\theta<2\pi$

But if $\theta=0$, then $r'=0$ and $\dfrac{dx}{d\theta}=\dfrac{dy}{d\theta}=0$

$$\lim_{\theta\to 0}\frac{dy}{dx}=\lim_{\theta\to 0}\frac{(1-\cos\theta)(1+2\cos\theta)}{\sin\theta(2\cos\theta-1)}=\lim_{\theta\to 0}\frac{(1+2\cos\theta)}{(2\cos\theta-1)}\lim_{\theta\to 0}\frac{(1-\cos\theta)}{\sin\theta}$$

$$=3\lim_{\theta\to 0}\frac{(1-\cos\theta)}{\sin\theta} \qquad \left[3\cdot\frac{0}{0}\right]$$

$$\overset{[H]}{=} \quad 3\lim_{\theta\to 0}\frac{\sin\theta}{\cos\theta}=3(0)=0$$

Therefore, equation of the tangent line is $\theta=0$

“graph”

Extra: Is there any way to distinguish between a dimpled limacon and a convex limacon, other than memorizing the relationship between *a* and *b*.

A cute method is to look at the <u>number</u> of (horizontal tangent lines.

(a) convex limacon ($a\ge 2b$) for example $r=4+2\sin\theta$

$$y=(4+2\sin\theta)\sin\theta=4\sin\theta+2\sin^2\theta$$

$\dfrac{dy}{dx}=4\cos\theta+4\sin\theta\cos\theta$. For a hor. tangent $\dfrac{dy}{dx}=0, \dfrac{dx}{d\theta}\ne 0$

$\therefore 0=4\cos\theta(1+\sin\theta)$

$\cos\theta=0$, $\sin\theta=-1$

$\theta=\dfrac{\pi}{2},\dfrac{3\pi}{2}$ or two tangent line

“graph”

(b) dimpled limacon $(b < a < 2b)$ for example $r = 3 + 2\sin\theta$

$$y = (3 + 2\sin\theta)\sin\theta = 3\sin\theta + 2\sin^2\theta$$

$$\frac{dy}{d\theta} = 3\cos\theta + 4\sin\theta\cos\theta$$

$$0 = \cos\theta(3 + 4\sin\theta)$$

$$\cos\theta = 0,\ \sin\theta = -\frac{3}{4}$$

$\theta = \frac{\pi}{2}, \frac{3\pi}{2}$ and one angle in the 3rd and one in the 4th quadrant.

$\therefore$ Therefore, there are four tangent lines.

"graph"