Power Series

Definition:

If $c_0, c_1, c_2, c_3, \ldots$ are constants and x is a variable, then a series of the form

$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \ldots + c_n x^n + \ldots$ is called a power series in *x*.

For each fixed value of *x*, the series is a series of constants that converges or diverges. A power series may converge for some values of *x* and diverges for other values of *x*.

Examples:

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \ldots$$ (also happens to be the geometric series with $r = x$)

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$$

Theorem.

For any power series in *x*, exactly one of the following is true:

(a) The series converges only for $x = 0$.

(b) The series converges for all real values of *x*.

(c) There is a positive number R such the series converges for some finite interval

$(-R, R)$, $[-R, R]$, $[-R, R)$, or $(-R, R]$.

The number *R* is called the radius of convergence of the power series. By agreement the radius of convergence in (a) is $R = 0$ and $R = \infty$ in (b). The interval of convergence of a power series is the interval that consists of all values of *x* for which the series converges. The interval of convergence in (a) is just a single point 0, in (b) it is $(-\infty, \infty)$ and in (c) it is one of the four intervals listed, depending what happens at the endpoints.

How do we find the radius and interval of convergence of a power series?

Examples.

1. $\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \ldots$ Using the ratio test we have

   $L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{x^{n+1}}{x^n} \right| = \lim_{n\to\infty} |x| = |x|$. Therefore, the series is absolutely convergent, and

   therefore convergent when $|x| < 1$. We must investigate convergence at the endpoints.

If $x = 1$ we have $\sum_{n=0}^{\infty} 1^n = 1+1+1+1+. \quad .$ which is obviously divergent.

If $x = -1$ we have $\sum_{n=0}^{\infty} (-1)^n = 1-1+1-1+...$ which is also divergent.

therefore, the interval of convergence is $(-1,1)$ with radius of convergence R = 1.

2. $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ Using the ratio test we have $L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n}\right| = \lim_{n\to\infty} \left|\frac{x}{n+1}\right| = 0$.

Therefore, the series is absolutely convergent, and therefore convergent for all real numbers. The interval of convergence is $(-\infty, \infty)$ and the radius of convergence is $R = \infty$.

3. $\sum_{n=0}^{\infty} n!x^n$ If $x \neq 0$ then the ratio test for absolute convergence yields

$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)!x^{n+1}}{n!x^n}\right| = \lim_{n\to\infty} |(n+1)x| = \infty$. Therefore, the series diverges for all nonzero values of x. The interval of convergence is $x = \{0\}$ and the radius of convergence is $R = 0$.

4. $\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{3^n (n+1)}$ Since $\left|(-1)^n\right| = \left|(-1)^{n+1}\right| = 1$ we use the ratio test to get

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{x^{n+1}}{3^{n+1}(n+2)} \cdot \frac{3^n (n+1)}{x^n}\right| = \lim_{n\to\infty} \left|\frac{|x|}{3} \cdot \left(\frac{n+1}{n+2}\right)\right| = \frac{|x|}{3} \lim_{n\to\infty} \left(\frac{1+\frac{1}{n}}{1+\frac{2}{n}}\right) = \frac{|x|}{3}$$

Therefore, the series converges absolutely when $\frac{|x|}{3} < 1$ or $|x| < 3$. Testing the endpoints yields:

When $x = -3$ then $\sum_{n=0}^{\infty} \frac{(-1)^n (-3)^n}{3^n (n+1)} = \sum_{n=0}^{\infty} \frac{(-1)^n (-1)^n 3^n}{3^n (n+1)} = \sum_{n=0}^{\infty} \frac{1}{n+1}$ which is the divergent harmonic series $1 + \frac{1}{2} + \frac{1}{3} + ...$

When $x = 3$ then $\sum_{n=0}^{\infty} \frac{(-1)^n 3^n}{3^n (n+1)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + ...$ which is the convergent alternating harmonic series.

Therefore, The interval of convergence is $(-3, 3]$ and the radius of convergence is $R = 3$.

Power Series in $x - a$.

If a is a constant and if x in the power series

$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \ldots + c_n x^n + \ldots$ is replaced by $x - a$ then the resulting series has the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + c_4(x-a)^4 + \ldots + c_n(x-a)^n + \ldots$$

This is called a power series in $x - a$.

Examples:

$$\sum_{n=0}^{\infty} \frac{(x-1)^n}{n+1} = 1 + \frac{(x-1)}{2} + \frac{(x-1)^2}{3} + \frac{(x-1)^3}{4} + \ldots \qquad \text{(a power series in } x-1\text{.)}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n (x+3)^n}{n!} = 1 - (x+3) + \frac{(x+3)^2}{2!} - \frac{(x+3)^3}{3!} + \ldots \qquad \text{(a power series in } x+3\text{.)}$$

The Power Series Theorem can be extended to $x - a$.

For any power series in $x - a$, exactly one of the following is true:

(a) The series converges only for $x = a$.

(b) The series converges for all real values of x.

(c) There is a positive number R such the series converges for some finite interval $(a-R, a+R)$, $[a-R, a+R]$, $[a-R, a+R)$, or $(a-R, a+R]$.

The power series in $x - a$ has its interval of convergence always centered at $x = a$.

Examples:

Find the interval of convergence and the radius of convergence of the power series.

(a) $\sum_{n=1}^{\infty} \frac{(x-5)^n}{n^2}$ (b) $\sum_{n=1}^{\infty} \frac{(x-1)^n}{n}$

Solution:

(a) Using the ratio test we have

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(x-5)^{n+1}}{(n+1)^2}\cdot\frac{n^2}{(x-5)^n}\right| = \lim_{n\to\infty}\left|(x-5)\left(\frac{n}{n+1}\right)^2\right| = |x-5|\lim_{n\to\infty}\left(\frac{1}{1+\frac{1}{n}}\right)^2 = |x-5|$$

. Therefore, the series is absolutely convergent, and therefore convergent if $|x-5| < 1$ or $-1 < x-5 < 1$, or $4 < x < 6$. Checking the endpoints yields:

If $x = 6$ then the series becomes $\sum_{n=1}^{\infty} \frac{1^n}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots$ which is a convergent p-series (p=2).

If $x = 4$ then the series becomes $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = -1 + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} - \ldots$ which converges absolutely and hence the series converges.

Therefore, the given series has an interval of convergence of [4,6] with a radius of convergence of R = 1.

(b) Using the ratio test we have

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(x-1)^{n+1}}{n+1}\cdot\frac{n}{(x-1)^n}\right| = \lim_{n\to\infty}\left|(x-1)\frac{n}{n+1}\right| = |x-1|\lim_{n\to\infty}\left(\frac{1}{1+\frac{1}{n}}\right) = |x-1|.$$

Therefore, the series is absolutely convergent, and therefore convergent if $|x-1| < 1$ or $-1 < x-1 < 1$, or $0 < x < 2$. Checking the endpoints yields:

If $x = 2$ then the series becomes $\sum_{n=1}^{\infty} \frac{1^n}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$ which is the divergent harmonic series.

If $x = 0$ then the series becomes $\sum_{n=1}^{\infty} \frac{(-1)^n}{n} = -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \ldots$ which converges absolutely since it is the alternating harmonic series.

Therefore, the given series has an interval of convergence of $[0,2)$ with a radius of convergence of R = 1.

Representation of Functions as Power Series

In the previous work we started with a power series and talked about its sum (convergence). Now we shift our viewpoint. We start with a function and attempt to find its power series representation.

In this section we look at $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots = \sum_{n=0}^{\infty} x^n \qquad |x| < 1$ a function as a sum of a power series.

Be careful, For $|x| > 1$,

$\frac{1}{1-x} \approx 1 + x + x^2 + x^3 + x^4$ (approximation) [Similar to: $1 \approx 0.99999$

$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \ldots$ (actual equality) $1 = 0.\overline{9}$]

N.B. The function $f(x) = \frac{1}{1-x}$ is just a restatement for $\sum_{n=0}^{\infty} x^n$.

The template $\frac{1}{1-\square}$ will be very important in this section.

Example

Express the function $f(x) = \frac{1}{1-x^4}$ as the sum of a power series and find the interval of convergence.

Solution:

$$f(x) = \frac{1}{1-x^4} = \sum_{n=0}^{\infty} \left(x^4\right)^n = \sum_{n=0}^{\infty} x^{4n} = 1 + x^4 + x^8 + x^{16} + \ldots$$

This geometric series converges when $|x^4| < 1$, or $x^4 < 1$, or $x < 1$.

Therefore the interval of convergence is $(-1,1)$.

Example:

Find a series representing the function $g(x) = \frac{1}{1+4x^2}$.

Solution:

Be careful of the sign

$$g(x) = \frac{1}{1+4x^2} = \frac{1}{1-(-(2x)^2)} = \sum_{n=0}^{\infty} \left(-(2x)^2\right)^n = \sum_{n=0}^{\infty} (-1)^n 4^n x^{2n}$$

Furthermore,

$|-(2x)^2| < 1$ so $4x^2 < 1$ therefore, $(2x-1)(2x+1) < 0$ with interval of convergence $\left(-\frac{1}{2}, \frac{1}{2}\right)$.

Example:

Find series representing the function $f(x) = \frac{2}{3-x}$.

Solution:

Make sure you rearrange to form the template $\frac{1}{1-\square}$

$$f(x) = \frac{2}{3-x} = \frac{2}{3}\left(\frac{1}{1-\frac{x}{3}}\right) = \frac{2}{3}\sum_{n=0}^{\infty}\left(\frac{x}{3}\right)^n \quad \text{or} \quad 2\sum_{n=0}^{\infty}\frac{1}{3^{n+1}}x^n$$

The series converges when $\left|\frac{x}{3}\right| < 1$, or $|x| < 3$ with interval of convergence (−3,3).

Question: If $f(x) = \frac{2}{3-x} = \frac{2}{3}\sum_{n=0}^{\infty}\left(\frac{x}{3}\right)^n$ How is it possible for a negative number

$f(4) = \frac{2}{3-4} = -2$ to be equal to a positive number

$$\frac{2}{3}\sum_{n=0}^{\infty}\left(\frac{4}{3}\right)^n = \frac{2}{3} + \frac{2}{3}\left(\frac{4}{3}\right) + \frac{2}{3}\left(\frac{4}{3}\right)^2 + \frac{2}{3}\left(\frac{4}{3}\right)^3 + \ldots ?$$

Answer: The function is not equal to the infinite series <u>outside</u> the interval of convergence. The number 4 is outside of (−3,3).

The idea is similar to any other function. A formula is <u>not</u> a function without its domain!

Example:

We can have extra x's in the function. Find a series representing the function $f(x) = \frac{2x^2}{3-x}$.

Solution:

$$f(x) = \frac{2x^2}{3-x} = x^2\left(\frac{2}{3-x}\right) = \left(x^2\right)\ 2\sum_{n=0}^{\infty}\frac{1}{3^{n+1}}x^n \quad \text{(by the previous question)}$$

$$= 2\sum_{n=0}^{\infty}\frac{1}{3^{n+1}}x^{n+2}$$

By changing the initial value of *n* we can write

$$f(x) = \frac{2x^2}{3-x} = 2\sum_{n=0}^{\infty}\frac{1}{3^{n+1}}x^{n+2} = 2\sum_{n=2}^{\infty}\frac{1}{3^{n-1}}x^n$$

with interval of convergence (−3,3).

Integration and differentiation of functions leads to integration and differentiation of power series, term-by-term. In such cases the interval of convergence may change at the endpoints. Therefore, in most of these questions you are asked to state the radius of convergence, which is preserved under integration and differentiation.

Some expressions that we have encountered similar to our template $\frac{1}{1-\square}$ are:

$$\int \frac{1}{1+x}\,dx = \ln|1+x| + c$$

$$\frac{d(\arctan x)}{dx} = \frac{1}{1+x^2}$$

$$\frac{d\left(\frac{1}{1-x}\right)}{dx} = \frac{1}{(1-x)^2}$$

Using integration and differentiation we can arrive at some functions and new series.

Example:

Express $f(x) = \frac{1}{(1-x)^2}$ as a power series. State its radius of convergence.

Solution:

We can use the geometric series $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots = \sum_{n=0}^{\infty} x^n$ and by differentiation arrive at $f(x) = \frac{1}{(1-x)^2}$.

Differentiating $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots = \sum_{n=0}^{\infty} x^n$ we get

$$f(x) = \frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \ldots = \sum_{n=1}^{\infty} nx^{n-1} \quad \text{or} \quad \sum_{n=0}^{\infty} (n+1)x^n \quad \text{with R=1}$$

Example:

Express $f(x) = \ln(x+1)$ as a power series. State its radius of convergence.

Solution:

For positive values of x $f(x) = \ln(x+1) = \int \frac{1}{1+x}dx$ which is the integral of the alternating geometric series $1 - x + x^2 - x^3 + \ldots$.

Therefore,

$$f(x) = \ln(x+1) = \int \frac{1}{1+x}\,dx = \int 1 - x + x^2 - x^3 + \ldots\, dx$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} + C$$

But $f(0) = \ln(1) = 0$/ Hence $C = 0$ Therefore,

$$\ln(x+1) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} \quad |x| < 1 \text{ i.e R=1.}$$

Example:

Express $f(x) = \tan^{-1}(x)$ as a power series. State its radius of convergence.

Solution:

$\tan^{-1} x = \int \frac{1}{1+x^2}\,dx$ which is a geometric series with $r = (-x^2)$. Therefore,

$$\tan^{-1} x = \int \frac{1}{1+x^2}\,dx = \int \left(1 - x^2 + x^4 - x^6 + \ldots\right) dx = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots + C$$

But $\tan^{-1} 0 = 0$ so $C = 0$ giving us $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$

With radius of convergence, $R = 1$ since $|-x| < 1$ yields $|x| < 1$.

[A useful recall: If

n represents all the integers
then 2*n* represents all even integers
while 2*n*+1 represents all odd integers.]

A last example uses partial fractions.

Example:

Express $f(x) = \frac{x+2}{2x^2 - x - 1}$ as a power series. State its interval of convergence.

Solution

$$f(x) = \frac{x+2}{2x^2 - x - 1} = \frac{x+2}{(2x+1)(x-1)} = \frac{A}{2x+1} + \frac{B}{x-1}$$

Solving for A and B

$$x+2=A(x-1)+B(2x+1)$$

Let $x=1$ then $3=3B$, so B=1

Let $x=-\frac{1}{2}$ then $\frac{3}{2}=-\frac{3}{2}A$, so $A=-1$

Therefore,

$$f\left(x\right)=\frac{x+2}{2x^2-x-1}=\frac{-1}{2x+1}+\frac{1}{x-1}=-1\left(\frac{1}{1-(-2x)}\right)-1\left(\frac{1}{1-x}\right)$$

$$=-\sum_{n=0}^{\infty}\left(-2x\right)^n-\sum_{n=0}^{\infty}x^n=-\sum_{n=0}^{\infty}\left(\left[\left(-2n\right)^n+1\right]x^n\right)$$

We represented f as the sum of two geometric series; the first converges on $\left(-\frac{1}{2},\frac{1}{2}\right)$ and the second converges on $\left(-1,1\right)$. Hence the sum function converges on $\left(-\frac{1}{2},\frac{1}{2}\right)$

[Functions can be added, as long as we pay attention to their domains. Similarly, the series, which are equivalent to these special functions, can also be added as long as we pay attention to their intervals of convergence.]