Reiman Sums and the Definite Integral

Useful Formulae: (pg. 383)

$$1+2+3+\cdots+n=\sum_{i=1}^{n} i=\frac{n(n+1)}{2}$$

$$1^2+2^2+3^2+\cdots+n^2=\sum_{i=1}^{n} i^2=\frac{n(n+1)(2n+1)}{6}$$

$$1^3+2^3+3^3+\cdots+n^3=\sum_{i=1}^{n} i^3=\left[\frac{n(n+1)}{2}\right]^2$$

$$\underbrace{1+1+1+\cdots+1}_{n}=\sum_{i=1}^{n} 1=n(1)=n$$

Example: Evaluate the Riemann sum for $f(x)=4x,\ 0\le x\le 2$ with four subintervals, taking the sample points to be

(a) left endpoints

(b) right endpoints

Solution:

"graph"

(a) The width of each subinterval is $\frac{2}{4}=\frac{1}{2}$

$$L_4=\frac{1}{2}f(0)+\frac{1}{2}f\left(\frac{1}{2}\right)+\frac{1}{2}f(1)+\frac{1}{2}f\left(\frac{3}{2}\right)$$

$$L_4=\frac{1}{2}\cdot 0+\frac{1}{2}\cdot 2+\frac{1}{2}\cdot 4+\frac{1}{2}\cdot 6$$

$$L_4=0+1+2+3=6$$

(b) The width of each subinterval is $\frac{2}{4}=\frac{1}{2}$

$$L_4=\frac{1}{2}f\left(\frac{1}{2}\right)+\frac{1}{2}f(1)+\frac{1}{2}f\left(\frac{3}{2}\right)+\frac{1}{2}f(2)$$

$$L_4=\frac{1}{2}\cdot 2+\frac{1}{2}\cdot 4+\frac{1}{2}\cdot 6+\frac{1}{2}\cdot 8$$

$$L_4=1+2+3+4=10$$

In Calculus 1500 you used the <u>definition</u> of a derivative,
$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ to find derivatives of various functions. Some of these were fairly time consuming. Similarly, we use the definition of a definite integral to find definite integrals. This process is also time consuming!

Definition:

$$\int_a^b f(x)\,dx=\lim_{n\to\infty}\sum_{i=1}^{n} f\left(x_i^*\right)\Delta x$$

Example: Use the <u>definition</u> of a definite integral to find $\int_0^2 4x\,dx$

Solution: (Using left endpoints) "sketch"

$$\int_0^2 4x\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f\left(x_i^*\right)\Delta x$$

$$= \lim_{n\to\infty}\sum_{i=1}^{n} 4\left(0+\frac{2(i-1)}{n}\right)\frac{2}{n}$$

$$= \lim_{n\to\infty}\frac{16}{n^2}\sum_{i=1}^{n}(i-1)$$

$$= \lim_{n\to\infty}\frac{16}{n^2}\left[\sum_{i=1}^{n} i - \sum_{i=1}^{n} 1\right]$$

$$= \lim_{n\to\infty}\frac{16}{n^2}\left[\frac{n(n+1)}{2}-n\right]$$

$$= \lim_{n\to\infty}\frac{16}{n^2}\left[\frac{n^2-n}{2}\right]$$

$$= \lim_{n\to\infty}\frac{8n^2-8n}{n^2} = \lim_{n\to\infty}\left(8-\frac{8}{n}\right) = 8$$

<u>Interval</u>	<u>L.E.</u>
1	0
2	$\frac{2}{n}$
3	$\frac{4}{n}$
i	$\frac{2(i-1)}{n}$

$\therefore \Delta x = \frac{2}{n}$

and $x_i^* = 0 + \frac{2(i-1)}{n}$

If we had used the right endpoints we would obtain the same results as illustrated below.

$$\int_0^2 4x\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f\left(x_i^*\right)\Delta x$$

$$= \lim_{n\to\infty}\sum_{i=1}^{n} 4\left(0+\frac{2i}{n}\right)\frac{2}{n}$$

$$= \lim_{n\to\infty}\frac{16}{n^2}\sum_{i=1}^{n}(i)$$

$$= \lim_{n\to\infty}\frac{16}{n^2}\left[\frac{n(n+1)}{2}\right]$$

$$= \lim_{n\to\infty}\frac{8}{n^2}\cdot\left(n^2+n\right)$$

$$= \lim_{n\to\infty}\frac{8n^2+8n}{n^2}$$

$$= \lim_{n\to\infty}\left(8+\frac{8}{n}\right) = 8$$

Interval	R.E.
1	$\frac{2}{n}$
2	$\frac{4}{n}$
3	$\frac{6}{n}$
i	$\frac{2i}{n}$

$\therefore \Delta x = \frac{2}{n}$

and $x_i^* = 0 + \frac{2i}{n}$

Let’s change the starting point.

Example: Use the <u>definition</u> of a definite integral to find $\int_1^2 4x\,dx$

Solution: (Using right endpoints.)

“sketch”

$$\int_1^2 4x\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f\left(x_i^*\right)\Delta x$$

$$= \lim_{n\to\infty}\sum_{i=1}^{n} 4\left(1+\frac{i}{n}\right)\frac{1}{n}$$

$$= \lim_{n\to\infty}\frac{4}{n}\sum_{i=1}^{n}\left(1+\frac{i}{n}\right)$$

$$= \lim_{n\to\infty}\frac{4}{n}\left[\sum_{i=1}^{n}1+\frac{1}{n}\sum_{i=1}^{n}i\right]$$

$$= \lim_{n\to\infty}\frac{4}{n}\cdot\left(n+\left(\frac{1}{n}\right)\frac{n(n+1)}{2}\right)$$

$$= \lim_{n\to\infty}\left[4+\frac{4n+4}{2n}\right]$$

$$= \lim_{n\to\infty}(4+2) = 6$$

<u>Interval</u>	<u>R.E.</u>
1	$1+\frac{1}{n}$
2	$1+\frac{2}{n}$
3	$1+\frac{3}{n}$
i	$1+\frac{i}{n}$

$\therefore \Delta x = \frac{1}{n}$

and $x_i^* = 1+\frac{i}{n}$

Let's change the function.

Example: Use the <u>definition</u> of a definite integral to find $\int_0^5 \left(25 - x^2\right) dx$

Solution: (Using right endpoints.)

"sketch"

$$\int_0^5 \left(25 - x^2\right) dx = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(x_i^*\right)\Delta x$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(25 - \left(\frac{5i}{n}\right)^2\right)\frac{5}{n}$$

$$= \lim_{n\to\infty} \frac{5}{n}\left(\sum_{i=1}^{n} 25 - \frac{25}{n^2}\sum_{i=1}^{n} i^2\right)$$

$$= \lim_{n\to\infty} \frac{5}{n}\left[25n - \frac{25}{n^2}\cdot\frac{n(n+1)(2n+1)}{6}\right]$$

$$= \lim_{n\to\infty} \left[125 - \frac{125}{6}\cdot\frac{2n^3 + 3n^2 + 1n}{n^3}\right]$$

$$= \lim_{n\to\infty} \left[125 - \frac{250}{6} - \frac{375}{6n} - \frac{125}{6n^2}\right]$$

$$= 125 - \frac{250}{6} = \frac{250}{3}$$

<u>Interval</u>	<u>R.E.</u>
1	$\frac{5}{n}$
2	$\frac{10}{n}$
3	$\frac{15}{n}$
i	$\frac{5i}{n}$

$\therefore \Delta x = \frac{5}{n}$

and $x_i^* = 0 + \frac{5i}{n}$

One last change.

Example: Use the <u>definition</u> of a definite integral to find $\int_2^5 \left(25 - x^2\right) dx$

Solution: (Using right endpoints.)

"sketch"

$$\int_2^5 \left(25 - x^2\right) dx = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(x_i^*\right)\Delta x \qquad \therefore \Delta x = \frac{3}{n} \text{ and } x_i^* = 2 + \frac{3i}{n}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(25 - \left(2 + \frac{3i}{n}\right)^2\right)\frac{3}{n}$$

$$= \lim_{n\to\infty} \frac{3}{n}\left(\sum_{i=1}^{n} 25 - \sum_{i=1}^{n}\left(4 + \frac{12i}{n} + \frac{9i^2}{n^2}\right)\right)$$

$$= \lim_{n\to\infty} \frac{3}{n}\left[25n - 4n - \frac{12}{n}\sum_{i=1}^{n} i - \frac{9}{n^2}\sum_{i=1}^{n} i^2\right]$$

$$= \lim_{n\to\infty} \frac{3}{n}\left[21n - \frac{12}{n}\cdot\frac{n(n+1)}{2} - \frac{9}{n^2}\cdot\frac{n(n+1)(2n+1)}{6}\right]$$

$$= \lim_{n\to\infty}\left[63 - \frac{18n+18}{n} - \frac{9}{2}\cdot\frac{2n^2+3n+1}{n^2}\right]$$

$$= 63 - 18 - 9 = 36$$

Some questions that are used to test your knowledge of the definition of a definite integral, but not so time consuming, follow.

Example: Use the <u>definition</u> of a definite integral to state $\int_1^5 \left(x^3 - \ln x\right) dx$ as a limit of a Rieman sum.

Solution: One solution is:

The length of each subinterval is $\frac{5-1}{n} = \frac{4}{n}$

Using right endpoints, the ith argument is: $1 + \frac{4i}{n}$

$$\therefore \int_1^5 \left(x^3 - \ln x\right)dx = \lim_{n\to\infty}\sum_{i=1}^{n}\left[\left(1+\frac{4i}{n}\right)^3 - \ln\left(1+\frac{4i}{n}\right)\right]\frac{4}{n}$$

Example: Write $\lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{\pi}{n}+e^{\frac{2\pi}{n}}\right)\frac{\pi}{n}$ as a definite integral. Evaluate this definite integral.

Solution:

$\Delta x = \frac{\pi}{n}$ and the right endpoint of i^{th} subinterval is $0+\frac{\pi}{n}$

$$\therefore \lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{\pi}{n}+e^{\frac{2\pi}{x}}\right)\frac{\pi}{n} = \int_0^{\pi}\left(x+e^{2x}\right)dx$$

$$= \left.\frac{x^2}{2}+\frac{e^{2x}}{2}\right|_0^{\pi} = \frac{\pi^2}{2}+\frac{e^{2\pi}}{2}-\left(0+\frac{1}{2}\right)$$

$$= \frac{e^{2\pi}+\pi^2-1}{2}$$

Example: (Variation of #68 pg. 393) Express $\lim_{n\to\infty}\frac{2}{n}\sum_{i=1}^{n}\frac{1}{1+\left(3+\frac{2i}{n}\right)^2}$ as a definite integral.

Solution: One solution is:

$\Delta x = \frac{2}{n}$ and the right endpoint of the i^{th} subinterval is $3+\frac{2i}{n}$

$$\therefore \lim_{n\to\infty}\frac{2}{n}\sum_{i=1}^{n}\frac{1}{1+\left(3+\frac{2i}{n}\right)^2} = \int_3^5 \frac{1}{1+x^2}dx$$