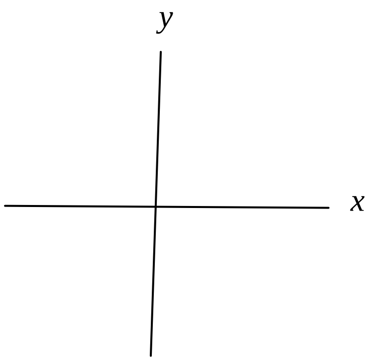
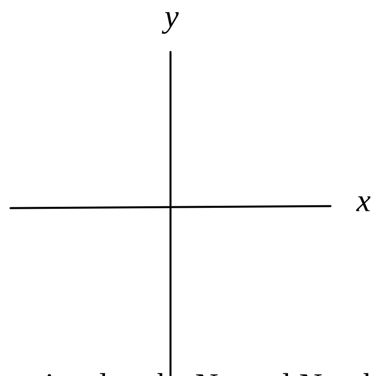
Sequences

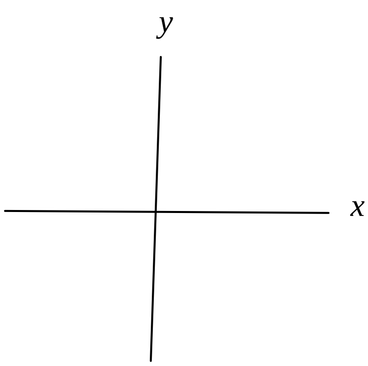
Sketch and state the domain of each of the following functions.

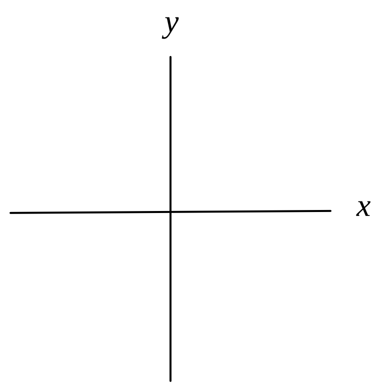
(a) $f(x) = x^2$ (b) $g(x) = \dfrac{x}{x+1}$





Sketch the above functions if the domains are restricted to the Natural Numbers.





Definition:
An infinite sequence is a function whose domain is the set of positive integers.
I.e. The function $f(x) = x^2$ becomes an infinite sequence if its domain is the set of positive integers. It is usually denoted by $a_n = n^2$ to indicate the natural umber domain.

Notation:

(a) $a_n = n^2$

or $\{n^2\}_1^\infty$

or $\{n^2\}$

(b) $b_n = \dfrac{n}{n+1}$

or $\left\{\dfrac{n}{n+1}\right\}_1^\infty$

or $\left\{\dfrac{n}{n+1}\right\}$

Note: Using 1, 3, 5, ... to denote the odd positive integers is BAD since the sequence could also denote the sequence 1, 3, 5, 11, 13,15, 21, 23, 25,...

Preferred notation for the odd positive integers is $\{2n-1\}$ or $\{2n-1\}_1^\infty$

Examples:
Write the first five terms of each sequence.

$(a)\quad \left\{\frac{n^2+2}{n^2+n+2}\right\}$ $\qquad (b)\quad \left\{\frac{1}{n}\right\}$ $\qquad (c)\quad \left\{3^n\right\}$

$(d)\quad \left\{\sqrt{n-3}\right\}_3^{\infty}$ $\qquad (e)\quad \left\{\cos(n\pi)\right\}$ $\qquad (f)\quad \left\{(-1)^{n+1}\right\}$

$\uparrow$

Required if domain is not all natural numbers

Examples:
Find an explicit n^{th} term definition for each infinite sequence. i.e. Find a formula for the general term a_n of the sequence. (Assume the most obvious pattern continues.)

$(a)\quad \frac{1}{2},\frac{2}{3},\frac{3}{4},\frac{4}{5},\ldots$

$(b)\quad \frac{2}{5},\frac{4}{8},\frac{6}{11},\frac{8}{14},\ldots$

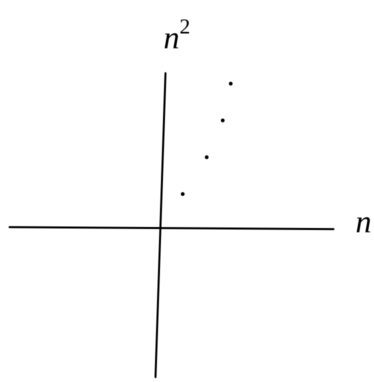
$(c)\quad \frac{2}{3},\frac{3}{9},\frac{4}{27},\frac{5}{81},\ldots$

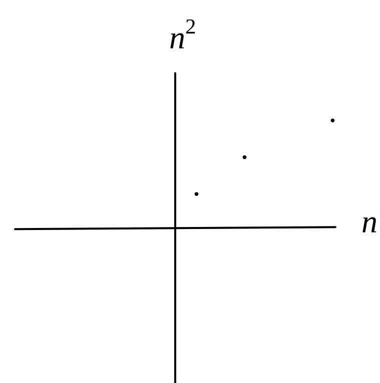
$(d)\quad -1,\frac{4}{5},-\frac{3}{5},\frac{8}{17},\ldots$

Limits of Sequences

How do the graphs of the following two sequences differ?

$\left\{n^2\right\}$ $\qquad\qquad \left\{\frac{n}{n+1}\right\}$





Diverges $\qquad\qquad$ Converges to the number 1

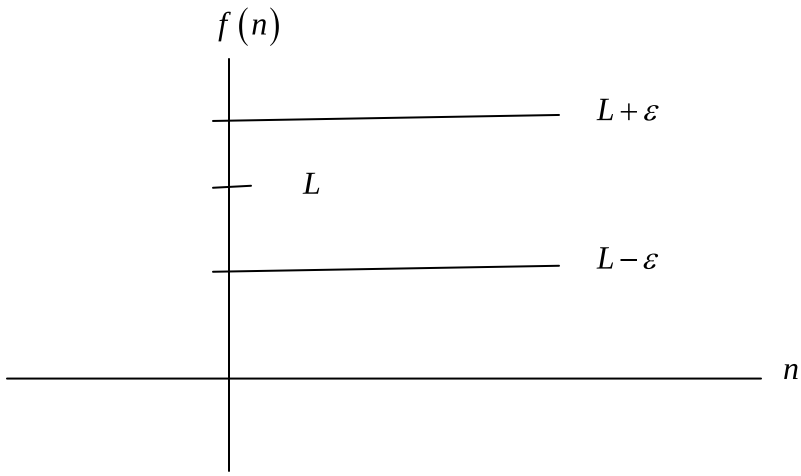
"Definition"

$\lim_{x\to\infty} c_n = L$ provided the values of c_n get closer and closer to L as $n \to \infty$

[$n \to \infty$ graphically means advancing far to the right.]

We say the sequence $\{c_n\}$ <u>converges</u> to L and $\{c_n\}$ is called a <u>convergent sequence</u>.

[Note: Only a finite number of the terms of $\{c_n\}$ may be outside the "lane" determined by $(L+\varepsilon, L-\varepsilon)$ for any value of ε.



If the number L does not exist then $\{c_n\}$ <u>diverges</u> and is called a <u>divergent sequence</u>.

This definition is a special case of the definition of a limit of a function where the domain is the set of natural numbers. Thus we have following theorem.

<u>Theorem</u>

If $\lim_{x\to\infty} f(x) = L$ and $f(n) = a_n$ where n is an integer, then $\lim_{n\to\infty} a_n = L$.

Example:

Since $\lim_{x\to\infty} \frac{x-2}{x} = 1$ it follows that $\lim_{n\to\infty} \frac{n-2}{n}$ is also 1, where $f(x) = \frac{x-2}{x}$ is the related, or associated function, to $a_n = \frac{n-2}{n}$.

Is the converse true? i.e. If $f(x)$ is the related function to $\{a_n\}$ and $\lim_{n\to\infty} a_n$ exists must $\lim_{x\to\infty} f(x)$ also exist? NO!

Look at the function $f(x) = \sin(2\pi x)$.

We can partition even further to get a very useful theorem.

Definition:

A <u>subsequence</u> of a sequence $\{a_n\}$ is a sequence whose terms are terms of the original sequence $\{a_n\}$ arranged in the same order.

i.e. A subsequence of a sequence $\{a_n\}$ has the form $a_{n_1}, a_{n_2}, a_{n_3}, \ldots$ where $n_1 < n_2 < n_3 < \ldots$ and is often denoted by $\{a_{n_k}\}$

For example:

The sequence $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$ is a subsequence of the sequence $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$

<u>Theorem</u>:

Every subsequence of a convergent sequence converges, and its limit is the limit of the original sequence.

Example:

Since $\left\{\frac{1}{2n}\right\}$ is a subsequence of $\left\{\frac{1}{n}\right\}$ and $\lim_{n\to\infty}\frac{1}{n} = 0$ it follows that $\lim_{n\to\infty}\frac{1}{2n}$ is also 0.

An important use of this theorem is if we have at least two convergent subsequences of an original sequence and they converge to two different limits, then the original sequence must diverge.

Example:

Consider the sequence 1, 0, 1, 0, 1, 0, ...

Its subsequence 1, 1, 1, ... obviously converges to 1 and the subsequence 0, 0, 0, ... obviously converges to 0. However, the original sequence is divergent since it oscillates form 1 to 0.

A method to show that a sequence is divergent all you have to do is produce two subsequences of this sequence which converge to two different limits.

The Limit Laws of functions also hold for convergent sequences.

<u>Theorems</u>: If $\{a_n\}$ and $\{b_n\}$ are convergent sequences with $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$, then

(a) $\lim_{n\to\infty}(a_n \pm b_n) =$

(b) $\lim_{n\to\infty}(ka_n) =$

(c) $\lim_{n\to\infty}(a_n b_n) =$

(d) $\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) =$

(e) $\lim_{n\to\infty}(a_n)^p =$

Examples:

Find the limit of each sequence, if the limit exists

(a) $\left\{\left(2-\frac{1}{n^4}\right)\left(\frac{n+3}{n+1}\right)\right\}$

(b) $\left\{\frac{2^n+1}{3^n-2}\right\}$

(c) $\left\{2^{\frac{1}{n}}+\frac{(-1)^n}{n^6}\right\}$

(d) $\left\{(n-1)\sqrt{\frac{n+}{2n}}\right\}$

Limits with Radicals (Techniques)

1. Find $\lim_{n\to\infty}\left\{\frac{n}{\sqrt{n^2+1}}\right\}$.
2. Find $\lim_{n\to\infty}\left\{\frac{\sqrt{3n^2+n+1}}{2n-1}\right\}$.
3. Find $\lim_{n\to\infty}\left\{\sqrt{n^2+n}-n\right\}$.

Some other useful theorems:

Theorem: (Limit Squeeze Theorem)

If $a_n \le c_n \le b_n$ for $n \ge n_0$ for a positive integer a_0, and $\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} b_n$, then $\lim_{n\to\infty} c_n = L$.

Theorem:

If $\lim_{n\to\infty}|a_n| = 0,$ then $\lim_{n\to\infty} a_n = 0.$

Theorem:

If $\lim_{n\to\infty} a_n = L$ and the function *f* is continuous at *L*, then $\lim_{n\to\infty} f(a_n) = f(L)$.

Example:

Find $\lim_{n\to\infty}\sin\left(\frac{\pi}{n}\right)$ [Since $f(x)=\sin x$ is a continuous function you can use the above theorem.]

Sometimes you are interested in establishing merely the existence of a limit.

Monotonic Sequences

Definitions:

(a) A sequence $\{a_n\}$ is increasing if $a_n < a_{n+1}$ for all $n \ge 1$.

(b) A sequence $\{a_n\}$ is decreasing if $a_n > a_{n+1}$ for all $n \ge 1$.

(c) A sequence $\{a_n\}$ is non-decreasing if $a_n \le a_{n+1}$ for all $n \ge 1$.

(d) A sequence $\{a_n\}$ is non-increasing if $a_n \ge a_{n+1}$ for all $n \ge 1$.

[Note: (c) and (d) are useful if some of the terms are equal. For example: 1, 1, 2, 3, 5, ...]

If a sequence satisfies any one of the above properties the sequence is said to be monotonic.

Method:

To prove that a sequence is monotonic you must show that $a_{n+1} - a_n$ is positive, or negative.

[Note: Using the related function *f* to the sequence $\{a_n\}$ show that $f'(x) > 0$ or $f'(x) < 0$.]

Example:

Show that $\left\{\dfrac{n^2+2}{n^2+n+2}\right\}$ is monotonic.

[Hint: It is a good idea to write the first few terms of the sequence to decide whether it is increasing or decreasing.]

Bounded

Definitions:

(a) U is an upper bound of $\{a_n\}$ iff. $a_n \le U$ for all $n \ge 1$.

(b) V is an lower bound of $\{a_n\}$ iff. $V \le a_n$ for all $n \ge 1$.

A bounded sequence is a sequence which has a lower and an upper bound.

Method

To show that a number U is an upper bound show that $U - a_n \ge 0$ for all $n \ge 1$.

To show that a number V is an lower bound show that $V - a_n \le 0$ for all $n \ge 1$.

Example:

Show that $\left\{\dfrac{n}{n+1}\right\}$ is bounded.

Example:

Determine whether $\left\{ \dfrac{n+3}{2n+7} \right\}$ is monotonic, bounded, and has a limit.

This last example leads us to a very important theorem.

Theorem:

A bounded monotonic sequence has a limit.

[This is an existence theorem. It does not tell us what the limit is, just that it exists.]

[Note: This theorem is often used with recursively defined sequences.]

Why is boundedness necessary?

$\left\{ n^2 \right\}$ is monotonic but not bounded and $\lim\limits_{n\to\infty} \left\{ n^2 \right\} = \infty$.

Why must the sequence be monotonic?

$\left\{ (-1)^{n+1} \right\}$ is bounded with $U = 1$ and $V = -1$ but it is not monotonic.

Question:

Can you give an example of a sequence that is monotonic, bounded above and below, but its limit does not exist?