<u>Sequences (addition to other notes)</u>

A. Since sequences are functions with domains restricted to the natural numbers the function limit theorems hold for convergent sequence. Therefore we use the function techniques with sequences.

Do the following sequences $\{a_n\}$ converge or diverge? Justify your answer.

1. $a_n = \dfrac{e^n}{3^n}$ 2. $a_n = \dfrac{(-1)^n + n}{(-1)^n - n}$ 3. $a_n = (-1)^n \sqrt{n}$ 4. $a_n = (n+1)^{\frac{1}{2}} - n^{\frac{1}{2}}$

5. $a_n = (-1)^n \dfrac{1}{\sqrt{n}}$ 6. $a_n = (-1)^{2n+1}$ 7. $a_n = (-1)^n \cos\left(\dfrac{\pi}{2}(n+1)\right)$ 8. $a_n = (-1)^n \sin\left(\dfrac{\pi}{2}(2n+1)\right)$

B. Does the sequence $a_n = \dfrac{1 + n\cos(2\pi n)}{n}$ converge or diverge? Does the associated function $f(x) = \dfrac{1 + x\cos(2\pi x)}{x}$ converge or diverge?

C. A theorem which is useful in discussing the convergence, or divergence, of some sequences is:

A sequences converges to a limit L iff its subsequences also converge to L.
(This theorem is often used to show the divergence of a sequence.)

Example:

The sequence $a_n = (-1)^n = -1, 1, -1, 1, -1, 1, -1, 1, \ldots$ diverges since the subsequence of the odd numbered terms: $-1, -1, -1, -1, \ldots$ converges to -1 while the subsequence of the even numbered terms:1, 1, 1, ... converges to 1.

Series

N.B. Be sure you know the difference between a sequence and the sequence of its partial sums. A series is convergent if its sequences of partial sums approaches a number.

Key Theorem

The geometric series $\sum_{n=1}^{\infty} ar^{n-1}$ converges if $|r| < 1$. Otherwise, it diverges.

If it converges the sum is $\frac{a}{1-r}$.

Hence,

1. The series $\sum_{n=1}^{\infty} \frac{5^n}{3^{n+1}}$ diverges with $r = \frac{5}{3} > 1$.

2. The series $\sum_{n=1}^{\infty} \frac{4 \cdot 5^n - 5 \cdot 4^n}{6^n}$ converges to 10 since it is the sum of two geometric series (justified by the limit laws of series) with $r = \frac{5}{6}$ and $r = \frac{4}{6}$ respectively.

Both less than 1.

Key Theorem (Test for divergence)

If $\lim_{n\to\infty} a_n \neq 0$ or does not exist than $\sum_{n=1}^{\infty} a_n$ is divergent.

Hence,

The series $\sum_{n=1}^{\infty} \frac{n}{n+1}$ is divergent since $\lim_{n\to\infty}\left(\frac{n}{n+1}\right)$ = 1 which is not 0.

Be careful. The converse is not true. $\lim_{n\to\infty}\left(\frac{1}{n}\right) = 0$ while the harmonic series $\sum_{n=1}^{\infty}\left(\frac{1}{n}\right)$ diverges.

The limit laws of sequences yield the limit laws of convergent series. Such a law was used in #2 above.

Is a series Convergent or Divergent?

You can use various tests to determine whether a series is convergent or divergent without necessarily finding the sum of the series. You did something of the sort when you used the comparison test for integral.

1. The Integral Test.

Given a sequence $\{a_n\}$ and its associated function $f(n) = a_n$ then if:

(a) f is continuous, positive and decreasing on $[1,\infty)$ and

(b) $\int_1^\infty f(x)\,dx$ is convergent (divergent)

then the series $\sum_{n=1}^{\infty} a_n$ is convergent (divergent).

Examples:

(a) The function $f(x) = \frac{1}{x}$ is continuous, positive and decreasing on $[1,\infty)$ and is divergent so by the integral test the series $\sum_{n=1}^{\infty} \frac{1}{n}$ is also divergent.

(b) By p-integral test $f(x) = \frac{1}{x^2}$, p=2, is continuous, positive, decreasing and convergent so by the integral test the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is also convergent.

This leads to a p-series test for series.

Theorem: The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p < 1$ and divergent if $p \geq 1$.

Therefore, (b) above could be answered <u>without resorting to the integral test</u>.

Example: Use the integral test to show that $\sum_{n=1}^{\infty} n^2 e^{-n^3}$ converges.

Solution: The function $f(x) = x^2 e^{-x^3}$ is continuous, positive and decreasing (since derivative $xe^{-x^3}(2-3x^3) < 0$ for $x \geq 1$) so the Integral Test applies.

Furthermore, $\int_1^\infty x^2 e^{-x^3}\,dx = \lim_{t\to\infty} \int_1^t x^2 e^{-x^3}\,dx = \lim_{t\to\infty} \left[-\frac{1}{3} x^{-x^3} \right]_1^t = -\frac{1}{3}\left(0 - \frac{1}{e}\right) = \frac{1}{3e}$

Since the improper integral is convergent then the associated series is also convergent by The Integral Test.

2. The Comparison Test (for series).

This test is similar to the integral comparison test applying to series whose terms are positive. If the terms of our series are SMALLER than the terms of a convergent series than our series also converges and if the terms of our series are GREATER than the terms of a divergent series than our series also diverges.

Theorem:

Suppose $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series with positive terms.

(a) If $\sum_{n=1}^{\infty} b_n$ is convergent and $a_n \le b_n$ for all n, then $\sum_{n=1}^{\infty} a_n$ is also convergent.

(b) If $\sum_{n=1}^{\infty} b_n$ is divergent and $a_n \ge b_n$ for all n, then $\sum_{n=1}^{\infty} a_n$ is also divergent.

Example:

Use the comparison test to show that $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ is convergent.

Solution:

For $n \ge 1$, $n^2+1 \ge n^2$ therefore, $\frac{1}{n^2+1} \le \frac{1}{n^2}$

But by p-series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent with p = 2.

Hence $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ is convergent by the series comparison test.

[Looks just like the integral comparison test. Doesn't it?]

N.B. As in the integral comparison test if your terms are greater than the terms of a convergent series the test does not apply. Similarly, if your terms are less than the terms of a divergent series the test again does not apply.

The series comparison test may be used to show that $\sum_{n=1}^{\infty} \frac{1}{2^n+1}$ is convergent by comparing this series to the convergent geometric series $\sum_{n=1}^{\infty} \frac{1}{2^n}$ since $2^n+1 \ge 2^n$.

However the comparison test will not work with $\sum_{n=1}^{\infty} \frac{1}{2^n-1}$ since $2^n-1 \le 2^n$.

We can get around this problem by using the following, very useful, Limit Comparison Test

Suppose $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series with positive terms. If ,

$\lim_{n\to\infty} \frac{a_n}{b_n} = \text{an positive finite number}$, then either both series converge or diverge.

Now we can handle our problem series $\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$.

$\lim_{n\to\infty} \frac{1/\left(2^n - 1\right)}{1/2^n} = \lim_{n\to\infty} \frac{2^n}{2^n - 1} = \lim_{n\to\infty} \frac{1}{1 - \frac{1}{2^n}} = 1$, a positive finite number and since the

geometric series $\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges it follows by The Limit Comparison Test that

$\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$ also converges.

Determine whether the series is convergent or divergent.

1. $\sum_{n=1}^{\infty} \frac{n}{n^2 + 1}$ 2. $\sum_{n=1}^{\infty} \frac{\ln n}{n^2 + 1}$ 3. $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n^9 - n^3 + 1}}$ 4. $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$

Use the Limit Comparison Test to determine if the series $\sum_{n=1}^{\infty} \frac{1}{3n - 2}$ converges or diverges.

[N.B. The difficulty in using the comparison test is the requirement of choosing a series to compare to your series. Other tests are often easier to apply.]

[More difficult.] Determine if the series converges or diverges.

1. $\frac{1}{2} + \frac{1}{2^3} + \frac{1}{2\cdot 3^2} + \frac{1}{4^3} + \frac{1}{2\cdot 5^2} + \frac{1}{6^3} + \ldots$

2. $\frac{1}{1} + \frac{1}{\ln 2} + \frac{1}{3} + \frac{1}{\ln 4} + \frac{1}{5} + \frac{1}{\ln 6} + \ldots$

3 . Alternating Series

How to deals with series whose terms are not necessarily positive.

Series whose terms alternate between positive and negative are called alternating series.

For example:

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\dots$$

or $$\sum_{n=1}^{\infty}(-1)^{n}\frac{1}{n}=-1+\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\dots$$

Generally alternating series have the form:

$$\sum_{n=1}^{\infty}(-1)^{n+1}a_n=a_1-a_2+a_3-a_4+\dots \quad \text{or} \quad \sum_{n=1}^{\infty}(-1)^{n}a_n=-a_1+a_2-a_3+a_4-\dots$$

The Alternate Series Test

This test has three conditions for series to be convergent

1. The series must alternate.
2. The terms must decrease (in absolute value) for large *n*.
3. The *n*th term must go to 0.

Use the alternating series test to show that each series converges.

1. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$ 2. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n+3}{n(n+1)}$

Solution:

1. The series is obviously alternating and $a_n=\frac{1}{n}>\frac{1}{n+1}=a_{n+1}$, hence decreasing.

Furthermore, $\lim_{n\to\infty}a_n=\lim_{n\to\infty}\frac{1}{n}=0$. By alternate series test $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$ converges.

2. The series is obviously alternating and

$$\frac{a_{n+1}}{a_n}=\frac{\dfrac{n+4}{(n+1)(n+2)}}{\dfrac{n+3}{(n)(n+1)}}=\frac{n(n+4)}{(n+3)(n+2)}=\frac{n^2+4n}{n^2+5n+6}<1 \text{ since } 4n<(5n+6).$$ Hence

$a_n>a_{n+1}$. [Notice the strategy to show that $a_n>a_{n+1}$]

Furthermore, $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n+3}{n(n+1)} = \lim_{n\to\infty} \frac{\frac{1}{n}+\frac{3}{n^2}}{1+\frac{1}{n}} = 0$. By alternate series test

$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n+3}{n(n+1)}$ converges.

Since the third requirement of the Alternating Series test is $\lim_{n\to\infty} a_n = 0$, is the second condition of $a_n > a_{n+1}$ really required. The answer is yes. We can construct a series in which the first and third conditions are satisfied but not the second. In the case where the series is $1-2+1 \ -\frac{1}{2}+1-\frac{1}{2} \ +\frac{1}{4}-\frac{1}{2}+\frac{1}{4} \ -\frac{1}{8}+\frac{1}{4}-\frac{1}{8} \ +\ldots$

The terms do go to 0, but they are not strictly decreasing. The partial sums go to 0 $1,-1,0,-\frac{1}{2},\frac{1}{2},0,\frac{1}{4},-\frac{1}{4},0,\frac{1}{8},-\frac{1}{8},0,\ldots$ so the series converges.

However the series $1-\frac{1}{2}+\frac{1}{2}-\frac{1}{4}+\frac{1}{3}-\frac{1}{8}+\frac{1}{4}-\frac{1}{16}+\frac{1}{5}-\frac{1}{32}+\ldots$ diverges since the positive terms form the harmonic series, and thus tend to infinity while the sum of the negative terms is -1.

Absolute Convergence and the Ratio and Root Tests

Some series are neither alternating nor a series of positive terms. The tests that follow can be applied to such series.

Definition:

A series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \ldots$ is said to converge absolutely if the series of absolute values $\sum_{n=1}^{\infty} |a_n| = |a_1| + |a_2| + |a_3| + \ldots$ converges. And

A series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \ldots$ is said to diverge absolutely if the series of absolute values $\sum_{n=1}^{\infty} |a_n| = |a_1| + |a_2| + |a_3| + \ldots$ diverges.

Example:
Determine whether the following series converges absolutely or diverges absolutely,

$(a)\ 1 - \frac{1}{2} - \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} - \frac{1}{2^5} - \ldots \qquad (b)\ 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \ldots$

Solution:

(a) The series of absolute values $1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \frac{1}{2^5} + \ldots$ is the convergent geometric series with $r = \frac{1}{2}$. Therefore, the given series converges absolutely.

(b) The series of absolute values $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ldots$ is the divergent harmonic series. Therefore, the given series diverges absolutely.

N.B. Theorem

If a series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, then it is convergent.

Example:

Determine whether the series $\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$ is convergent.

Solution:

We apply the Comparison Test to the series of absolute values, $\sum_{n=1}^{\infty} \left| \frac{\cos n}{n^2} \right|$.

$$\sum_{n=1}^{\infty}\left|\frac{\cos n}{n^2}\right| = \sum_{n=1}^{\infty}\frac{|\cos n|}{n^2} \le \sum_{n=1}^{\infty}\frac{1}{n^2} \text{ since } |\cos n| \le 1 \text{ for all } n.$$

But $\sum_{n=1}^{\infty}\frac{1}{n^2}$ is convergent (p-series with p=2>1) so by Comparison Test $\sum_{n=1}^{\infty}\frac{|\cos n|}{n^2}$ converges. Hence, $\sum_{n=1}^{\infty}\frac{\cos n}{n^2}$ is absolutely convergent and therefore $\sum_{n=1}^{\infty}\frac{\cos n}{n^2}$ converges.

Note: A convergent series that is <u>not absolutely</u> convergent is said to be <u>conditionally convergent</u>.

Hence, since the harmonic series $\sum_{n=1}^{\infty}\frac{1}{n}$ is divergent while the alternating harmonic series $\sum_{n=1}^{\infty}(-1)^{n-1}\left(\frac{1}{n}\right)$ is convergent we say that $\sum_{n=1}^{\infty}(-1)^{n-1}\left(\frac{1}{n}\right)$ is conditionally convergent.

Although we cannot generally infer convergence or divergence of a series from <u>absolute divergence</u>, the following test is very useful in determining whether a given absolute divergent series is divergent.

<u>The Ratio Test</u>

For a given series $\sum_{n=1}^{\infty} a_n$ if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L$ then if:

(a) L < 1 the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent and thus convergent.

(b) L > 1 or $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty$ the series diverges.

(c) L = 1 the Ratio Test fails. No conclusion can be drawn.

Example:

Use The Ratio Test to determine whether the series converges or diverges.

$(a)\quad \sum_{n=1}^{\infty}(-1)^n\frac{2^n}{n!} \qquad (b)\quad \sum_{n=1}^{\infty}(-1)^n\frac{(2n-1)}{3^n}$

Solution:

$(a)\qquad |a_n| = \left|(-1)^n \frac{2^n}{n!}\right| = \frac{2^n}{n!}$ Therefore,

$$L = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty}\left[\frac{2^{n+1}}{(n+1)!}\cdot\frac{n!}{2^n}\right] = \lim_{n\to\infty}\frac{2}{n+1} = 0 < 1.$$

Hence the series converges absolutely and therefore the series converges.

$(b)\qquad |a_n| = \left|(-1)^n \frac{(2n-1)!}{3^n}\right| = \frac{(2n-1)!}{3^n}$

$$L = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty}\left[\frac{(2(n+1)-1)}{3^{n+1}}\cdot\frac{3^n}{(2n-1)!}\right] = \lim_{n\to\infty}\frac{1}{3}\frac{(2n+1)!}{(2n-1)!} = \frac{1}{3}\lim_{n\to\infty}(2n)(2n+1) = \infty$$

which implies that the series diverges.

The following test is convenient to apply when <u>the *n*th power</u> occurs.

<u>The Root Test</u>

For a given series $\sum_{n=1}^{\infty} a_n$ if $\lim_{n\to\infty}\sqrt[n]{|a_n|} = L$ then if:

(a) L < 1 the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent and thus convergent.

(b) L > 1 or $\lim_{n\to\infty}\sqrt[n]{|a_n|} = \infty$ the series diverges.

(c) L = 1 the Root Test fails. No conclusion can be drawn.

<u>Example:</u>

Use The Root Test to determine whether the series converges or diverges.

(a) $\sum_{n=1}^{\infty}\frac{n^n}{n!}$ (b) $\sum_{n=1}^{\infty}\left(\frac{2n+5}{3n+1}\right)^n$

Solution:

(a) For the series $\sum_{n=1}^{\infty}\frac{n^n}{n!}$, $a_n = \frac{n^n}{n!}$

Therefore, $\sqrt[n]{|a_n|} = \sqrt[n]{\frac{n^n}{n!}} = \frac{n}{\sqrt[n]{n!}} > 1$ since $n > \sqrt[n]{n!}$

Hence, the series diverges.

(b)For the series $\sum_{n=1}^{\infty}\left(\frac{2n+5}{3n+1}\right)^n$, $a_n = \left(\frac{2n+5}{3n+1}\right)^n$ so

$$\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\sqrt[n]{\left(\frac{2n+5}{3n+1}\right)^n} = \lim_{n\to\infty}\frac{2n+5}{3n+1} = \lim_{n\to\infty}\frac{2+\frac{5}{n}}{3+\frac{1}{n}} = \frac{2}{3} < 1$$

Therefore the given series converges by The Root Test.

Summary of Convergence Tests

Test Name — Statement(will use $\sum a_n$ to denote $\sum_{n=1}^{\infty} a_n$) — Suggestions

Divergence Test	If $\lim_{n\to\infty} a_n \neq 0$, then $\sum a_n$ diverges	If $\lim_{n\to\infty} a_n = 0$, then $\sum a_n$ may or may not converge.
Integral Test	Let $\sum a_n$ be a series of positive terms. If f is a function that is decreasing and continuous on $[c,\infty)$ and such that $a_i = f(i)$ for all $n \geq c$, then $\sum_{n=1}^{\infty} a_n$ and $\int_c^{\infty} f(x)\,dx$ both converge or both diverge	Applies only to series of positive terms. Try this test when $f(x)$ is easy to integrate.
Comparison Test	Suppose $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series with positive terms. (a) If $\sum_{n=1}^{\infty} b_n$ is convergent and $a_n \leq b_n$ for all n, then $\sum_{n=1}^{\infty} a_n$ is also convergent. (b) If $\sum_{n=1}^{\infty} b_n$ is divergent and $a_n \geq b_n$ for all n, then $\sum_{n=1}^{\infty} a_n$ is also divergent.	Applies only to series of nonnegative terms. Try this as a last resort; other tests are often easier to apply.
Limit Comparison Test	Suppose $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series with positive terms. If , $\lim_{n\to\infty}\frac{a_n}{b_n} =$ an positive finite number , then either both series converge or diverge.	This is easier to apply than The Comparison Test, but still requires some skill in choosing the series $\sum_{n=1}^{\infty} b_n$ for comparison.

Ratio Test	For a given series $\sum_{n=1}^{\infty} a_n$ if $\lim_{n\to\infty}\left\|\frac{a_{n+1}}{a_n}\right\| = L$ then if: (a) L < 1 the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent and thus convergent. (b) L > 1 or $\lim_{n\to\infty}\left\|\frac{a_{n+1}}{a_n}\right\| = \infty$ the series diverges. (c) L = 1 the Ratio Test fails. No conclusion can be drawn.	Try this test when a_n involves factorials or *n*th powers. The series need not have positive terms and need not be an alternating series to use this test.
Root Test	For a given series $\sum_{n=1}^{\infty} a_n$ if $\lim_{n\to\infty}\sqrt[n]{\|a_n\|} = L$ then if: (a) L < 1 the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent and thus convergent. (b) L > 1 or $\lim_{n\to\infty}\sqrt[n]{\|a_n\|} = \infty$ the series diverges. (c) L = 1 the Root Test fails. No conclusion can be drawn.	Try this test when a_n involves *n*th powers.
Alternate Series Test	This test has three conditions for series to be convergent 1. The series must alternate. 2. The terms must decrease (in absolute value) for large *n*. 3. The $\lim_{n\to\infty} a_n = 0$.	This test applies only to alternating series.

This is a good time to read the strategies of testing series and do the accompanying exercise!