<u>Solutions to Parametric Area</u>

1.

(a) $y = 2t^2 + 1 > 0$ $\quad \dfrac{dx}{dt} = 3t^2 \geq 0$ $\quad$ "*sketch*"

$$\therefore A = \int_{-1}^{1} y\frac{dx}{dt}\,dt = \int_{-1}^{1}\left(2t^2+1\right)3t^2dt = \int_{-1}^{1} 6t^4 + 3t^2dt$$

$$= \frac{6t^5}{5} + t^3\Big|_{-1}^{1} = \frac{6}{5} + 1 - \left(\frac{6}{5} - 1\right) = \frac{22}{5}$$

(b) $y = e^{-t} > 0$ $\quad \dfrac{dx}{dt} = 3e^{3t} > 0$ $\quad$ "*sketch*"

$$\therefore A = \int_{0}^{\ln 2} y\frac{dx}{dt}\,dt = \int_{0}^{\ln 2} e^{-t}\left(3e^{3t}\right)dt = \int_{0}^{\ln 2}\left(3e^{2t}\right)dt$$

$$= \frac{3e^{2t}}{2}\Big|_{0}^{\ln 2} = \frac{3\left(e^{\ln 2}\right)^2}{2} - \frac{3\left(e^{\ln 0}\right)^2}{2} = \frac{3(2)^2}{2} - \frac{3(1)^2}{2} = 6 - \frac{3}{2} = \frac{9}{2}$$

(c) $y = \sin^2 t \geq 0$ $\quad \dfrac{dx}{dt} = -\sin t \leq 0$ $\quad [0, \pi]$ $\quad$ "*sketch*"

$$\therefore A = -\int_{0}^{\pi} \sin^2 t\left(-\sin t\right)dt = \int_{0}^{\pi} \sin^2 t \sin t dt = \int_{0}^{\pi}\left(1 - \cos^2 t\right)\sin t dt$$

$$= \int_{0}^{\pi} \sin t dt - \int_{0}^{\pi} \cos^2 t \sin t dt = -\cos t\Big|_{0}^{\pi} + \frac{\cos^3 t}{3}\Big|_{0}^{\pi}$$

$$= -\left[-1 - (1)\right] + \left[-\frac{1}{3} - \frac{1}{3}\right] = \frac{4}{3}$$

(d) $y = 2t + 1 > 0$ on $[0,1]$ $\quad \dfrac{dx}{dt} = -e^{t} < 0$ $\quad$ "*sketch*"

$$\therefore A = -\int_{0}^{1} y\frac{dx}{dt}\,dt = -\int_{0}^{1}(2t+1)\left(-e^{t}\right)dt = 2\int_{0}^{1}\left(te^{t}\right)dt + \int_{0}^{1} e^{t}\,dt$$

$$Let\;\; u = t \qquad dv = e^{t}dt$$

$$du = dt \qquad v = e^{t}$$

$$A = 2\left[te^{t}\Big|_{0}^{1} - \int_{0}^{1} e^{t}dt\right] + \int_{0}^{1} e^{t}dt = 2te^{t}\Big|_{0}^{1} - \int_{0}^{1} e^{t}dt = 2te^{t}\Big|_{0}^{1} - e^{t}\Big|_{0}^{1}$$

$$= 2e - e + 1 = e + 1$$

(e) $\quad y=0 \quad o=t(t-2)$ curve intersects x-axis at 0 and 2

$y=t^2-2t\le 0 \qquad \frac{dx}{dt}=2t\ge 0$ on $[0,2]$ "$sketch$"

$$\therefore A=-\int_0^2 y\frac{dx}{dt}dt=-\int_0^2\left(t^2-2t\right)2t\,dt=-\int_0^2 2t^3-4t^2dt$$

$$=-\frac{t^4}{2}+\frac{4t^3}{3}\Big|_0^2=-8+\frac{32}{3}-0=\frac{8}{3}$$

(f) $\quad y=0 \quad 0=t^2+t-2$ "$sketch$"

$0=(\mathrm{t}+2)(t-1)$ curve intersects x-axis at -2 and 1

Furthermore, $\frac{dx}{dt}=2t+2$

$0=2(t+1)$

$t=-1$

t	-2	-1	1
y	$-$		$-$
$\frac{dx}{dt}$	$-$		$+$
$product$	$+$		$-$

$On[-2,-1]$ *yields area of* B. $On[-1,1]$ *yields required area and* $(-$*area of* $B)$

Better: Counterclockwise closed curve $\therefore\ -Area$

$$\therefore A=-\int_{-2}^1 y\frac{dx}{dt}dt=-\int_{-2}^1\left(t^2+t-2\right)(2t+2)dt=-\int_{-2}^1 2t^3+4t^2-2t-4dt$$

$$=-\frac{t^4}{2}-\frac{4t^3}{3}+t^2+4t\Big|_{-2}^1=-\frac{1}{2}-\frac{4}{3}+1+4\left(-8+\frac{32}{3}+4-8\right)=\frac{9}{2}$$

2. (a) In the first quadrant $y=5\sin t\ge 0$ and $\frac{dx}{dt}=-4\cos t\le 0$ "$sketch$"

$$\therefore A=-\int_0^{\frac{\pi}{2}}5\sin t(-4\sin t)\,dt=20\int_0^{\frac{\pi}{2}}\sin^2 t dt=20\int_0^{\frac{\pi}{2}}\frac{1-\cos 2t}{2}dt$$

$$=20\left[\frac{1}{2}t-\frac{\sin 2t}{2}\right]_0^{\frac{\pi}{2}}=5\pi-0-[0]=5\pi$$

(b) In the first quadrant $y = 5\sin t \geq 0$

t	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$
y	$+$	$+$	
$\frac{dx}{dt}$	$+$	$-$	
product	$+$	$-$	

"*sketch*"

and $\dfrac{dx}{dt} = 8\cos 2t$

$0 = 8\cos 2t$

$2t = \dfrac{\pi}{2}, \dfrac{3\pi}{2}, \ldots$

$t = \dfrac{\pi}{4}$ in quadrant I *OR Counterclockwise*

$$\therefore A = -\int_0^{\frac{\pi}{2}} 5\sin t\left(8\cos 2t\right)dt \leftarrow \textit{area A gets added and then subtracted}.$$

$$= -40\int_0^{\frac{\pi}{2}} \sin t\left(2\cos^2 t - 1\right)dt = 80\left.\frac{\cos^3 t}{3}\right|_0^{\frac{\pi}{2}} - 40\cos t\Big|_0^{\frac{\pi}{2}}$$

$$= 0 - \frac{80}{3} - \left(0 - 40\right) = \frac{40}{3}$$

(c) In the first quadrant $y = e^t - 1 \geq 0$ and $\dfrac{dx}{dt} = -\sin t \leq 0$ "*sketch*"

$$\therefore A = -\int_0^{\frac{\pi}{2}} \left(e^t - 1\right)\left(-\sin t\right)dt = \int_0^{\frac{\pi}{2}} \left(e^t \sin t\right) + \left(-\sin t\right)dt$$

Using integration by parts to find $\int_0^{\frac{\pi}{2}} \left(e^t \sin t\right)dt$ we get

$$= \left[\frac{e^t \sin t}{2} - \frac{e^t \cos t}{2} + \cos t\right]_0^{\frac{\pi}{2}} = \frac{e^{\frac{\pi}{2}}}{2} - \left[-\frac{1}{2} + 1\right] = \frac{e^{\frac{\pi}{2}} - 1}{2}$$

3. (a) $y=0$ $0=3t^2-9$ "*sketch*"

$0=3(t+\sqrt{3})(t-\sqrt{3})$ curve intersects x-axis at $t=\pm\sqrt{3}$

Furthermore, $\frac{dx}{dt}=3t^2-3$

$0=3(t+1)(t-1)$

$t=\pm1$

t	$-\sqrt{3}$	-1	1	$\sqrt{3}$
y	$-$		$-$	$-$
$\frac{dx}{dt}$	$+$		$-$	$+$
product	$-$		$+$	$-$

On $[-\sqrt{3},\sqrt{3}]$ *areas of A and B get added and subtracted.*

Better: Clockwise closed curve $\therefore$ *Area is positve.*

$$\therefore A=\int_{-\sqrt{3}}^{\sqrt{3}} y\frac{dx}{dt}dt=\int_{-\sqrt{3}}^{\sqrt{3}}(3t^2-9)(3t^2-3)dt=2\int_{-\sqrt{3}}^{0}(9t^4-36t^2+27)dt$$

$$=18\int_{-\sqrt{3}}^{0}(t^4-4t^2+3)dt=18\left[\frac{t^5}{5}-\frac{4t^3}{3}+3t\right]_{-\sqrt{3}}^{0}=\frac{72\sqrt{3}}{5}$$

(b) $y=0$ $0=t^3-4t$ "*sketch*"

$0=t(t-2)(t+2)$ curve intersects x-axis at $t=0,\pm2$

Furthermore, $\frac{dx}{dt}=2t$

$0=2t$

$t=0$

t	-2	0	2
y	$+$		$-$
$\frac{dx}{dt}$	$-$		$+$
product	$-$		$-$

Better: Counterclockwise closed curve $\therefore$ *Area is negative.*

$$\therefore A=-\int_{-2}^{2} y\frac{dx}{dt}dt=\int_{-2}^{2}(t^3-4t)2tdt=2\int_{0}^{2}(2t^4-8t^2)dt$$

$$=-2\left[\frac{2t^5}{5}-\frac{8t^3}{3}\right]_0^2=-2\left[\frac{64}{5}-\frac{64}{3}-(0)\right]=\frac{256}{15}$$

(c) $y=0$ $0=t^3-3t$ "*sketch*"

$0=t\left(t+\sqrt{3}\right)\left(t-\sqrt{3}\right)$ curve intersects x-axis at $t=\pm\sqrt{3},0$

Recall: Curve intersected itself at $t=-1,2$

Furthermore, $\dfrac{dx}{dt}=2t-1$

$0=2t-1$

$t=\dfrac{1}{2}$

t	-1	0	$\frac{1}{2}$	$\sqrt{3}$	2
y	$+$	$-$	$-$	$+$	
$\frac{dx}{dt}$	$-$	$-$	$+$	$+$	
product	$-$	$+$	$-$	$+$	

The area is accumulated as follows:

Interval of t:	$(-1,0)$	$\left(0,\frac{1}{2}\right)$	$\left(\frac{1}{2},\sqrt{3}\right)$	$\left(\sqrt{3},2\right)$
	$-A-C$	$+B$	$-B-E-D$	A

Result $=-C-E-D$

<u>Better</u>: Counterclockwise closed curve $\therefore$ *Area is negative.*

$$\therefore A=-\int_{-1}^{2}y\frac{dx}{dt}\,dt=-\int_{-1}^{2}\left(t^3-3t\right)\left(2t-1\right)dt=-\int_{-1}^{2}\left(2t^4-t^3-6t^2+3t\right)dt$$

$$=-\left[\frac{2t^5}{5}-\frac{t^4}{4}-2t^3+\frac{3t^2}{2}\right]_{-1}^{2}=-\left[\frac{64}{5}-4-16+6\right]+\left[-\frac{2}{5}-\frac{1}{4}+2+\frac{3}{2}\right]=\frac{81}{20}$$

(d) $y=0 \quad 0=3t^2-3t$ "*sketch*"

$0=t\left(t+\sqrt{3}\right)\left(t-\sqrt{3}\right)$ curve intersects x-axis at $t=\pm\sqrt{3},0$

Recall: The curve crosses itself at $t=-1,2$.

Furthermore, $\dfrac{dx}{dt}=3t^2-6t$

$0=3t(t-2)$

$t=2,0$

t	-1	0	$\sqrt{3}$	2
y	$+$	$-$	$+$	
$\dfrac{dx}{dt}$	$+$	$-$	$-$	
product	$+$	$+$	$-$	

On $[-1,2]$ *area of A is added and subtracted.*

<u>Better</u>: Clockwise closed curve $\therefore$ *Area is positve.*

$$\therefore A=\int_{-1}^{2} y\frac{dx}{dt}\,dt=\int_{-1}^{2}\left(t^3-3t\right)\left(3t^2-6t\right)dt=\int_{-1}^{2}\left(3t^5-6t^4-9t^3+18t^2\right)dt$$

$$=\left[\frac{t^6}{2}-\frac{6t^5}{5}-\frac{9t^4}{4}+6t^3\right]_{-1}^{2}=32-\frac{192}{5}-36+48-\left(\frac{1}{2}+\frac{6}{5}-\frac{9}{4}-6\right)=\frac{243}{2}\quad 0$$