<u>Substitution</u>

Recall the chain rule from Calculus 1500

$\underline{f(x)}$	$\underline{f'(x)}$
$\left(x^3+x^2\right)^5$	$5\left(x^3+x^2\right)^4\left(3x^2+2x\right)$
$\sin^3 x$	$3\sin^2 x\cos x$
$e^{x^2+\sin x}$	$e^{x^2+\sin x}\left(2x+\cos x\right)$

When we integrate we are interesting in going backwards.

ie. $\int f\left(g(x)\right)g'(x)\,dx = f\left(g(x)\right)+C$

Even more generally, $\int f^n\left(g(x)\right)g'(x)\,dx = \dfrac{f^{n+1}\left(g(x)\right)}{n+1}+C$

Therefore,

$$\int\left(x^3+x^2\right)^4\left(3x^2+2x\right)dx = \frac{\left(x^3+x^2\right)^5}{5}+C$$

$$\int \sin^2 x\cos x\,dx = \frac{1}{3}\int 3\sin^2 x\cos x\,dx = \frac{\sin^3 x}{3}+C$$

$$\int e^{x^2+\sin x}\left(2x+\cos x\right)dx = e^{x^2+\sin x}+C$$

Generally look for $\boxed{\int\left[f(x)\right]^n f'(x)\,dx}$ and if $n=-1$ then

$$\boxed{\int \frac{f'(x)}{f(x)}dx = \int\left[f(x)\right]^{-1} f'(x)\,dx = \ln\left(f(x)\right)+C}$$

<u>Using substitution</u>

(a) $\int e^{x^2+\sin x}\left(2x+\cos x\right)dx$ $\qquad$ *Let* $u = x^2+\sin x$ *then* $\dfrac{du}{dx} = 2x+\cos x$

$= \int e^u du = e^u + C$ $\qquad$ *usually written as* $du = \left(2x+\cos x\right)dx$

$= e^{x^2+\sin x} + C$

(b) $\int \sin^2 x \cos x\, dx$ $\qquad$ *Let* $u = \sin x$ *then* $du = \cos x\, dx$

$= \int u^2\, du = \dfrac{u^3}{3} + C$

$= \dfrac{\sin^3 x}{3} + C$

(c) $\int \left(x^3+x^2\right)^4 \left(3x^2+2x\right)d \quad x$ $\qquad$ *Let* $u = x^3+x^2$

$= \int u^4\, du$ $\qquad$ *then* $du = \left(3x^2+2x\right)dx$

$= \dfrac{u^5}{5} + C$

$= \dfrac{\left(x^3+x^2\right)^5}{5} + C$

Integrals requiring substitution

Example: Find $\int x\left(x+5\right)^{100} dx$

Solution: The exponent is with the binomial-BAD. The integral . $\int x^{100}\left(x+5\right)dx$ would be so much easier to evaluate.

$\int x\left(x+5\right)^{100} dx$ $\qquad$ *Let* $u = x+5$ $\quad$ *then* $du = dx$

$= \int \left(u-5\right)u^{100}\, du$ ← We have turned the high exponent term into a monomial.

$= \int \left(\mathrm{u}^{101} - 5u^{100}\right)du = \dfrac{u^{102}}{102} - \dfrac{5u^{101}}{101} + C = \dfrac{\left(x+5\right)^{102}}{102} - \dfrac{5\left(x+5\right)^{101}}{101} + C$

Example: Evaluate: $\int x\sqrt{x+5}\,dx$

Solution:

Method (a)

$$\int x\sqrt{x+5}\,dx = \int x\left(x+5\right)^{\frac{1}{2}}\,dx$$

[Note: The exponent $\frac{1}{2}$ is the same problem as a very large exponent since the expansion of $\left(x+5\right)^{\frac{1}{2}}$ would contain an infinite number of terms.]

$Let\, u = x+5 \quad then\ du = dx$

$$\int x\sqrt{x+5}\,dx = \int x\left(x+5\right)^{\frac{1}{2}}\,dx = \int \left(u-5\right)u^{\frac{1}{2}}du$$

$$= \int \left(u^{\frac{3}{2}} - 5u^{\frac{1}{2}}\right)du = \frac{2u^{\frac{5}{2}}}{5} - 5u^{\frac{3}{2}} \cdot \frac{2}{3} + C$$

$$= \frac{2}{5}\left(x+5\right)^{\frac{5}{2}} - \frac{10}{3}\left(x+5\right)^{\frac{3}{2}} + C$$

Method (b) – avoids the fractional exponents until the very end.

$$\int x\sqrt{x+5}\,dx \qquad Let\ u = \sqrt{x+5} \quad then\ u^2 = x+5$$

$$= \int \left(u^2-5\right)u\left(2u\right)du \qquad and\ 2u\,du = dx$$

$$= \int \left(2u^4 - 10u^2\right)du$$

$$= \frac{2}{5}u^5 - \frac{10}{3}u^3 + C$$

$$= \frac{2}{5}\left(\sqrt{x+5}\right)^5 - \frac{10}{3}\left(\sqrt{x+5}\right)^3 + C \ \ or\ \ \frac{2}{5}\left(x+5\right)^{\frac{5}{2}} - \frac{10}{3}\left(x+5\right)^{\frac{3}{2}} + C$$

<u>Unnecessary substitution, but substitution nonetheless</u>

Example: Evaluate $\int \cos x \sin^3 x\,dx$

Solution: $\sin x$ is the function, while $\cos x$ is its derivative.

(a) substitution method.

$Let\ u = \sin x\ \ then\ du = \cos x\, dx$

$$\therefore \int \sin^3 x \cos x\, dx = \int u^3\, du = \frac{u^4}{4} + C = \frac{\sin^4 x}{4} + C$$

(b) nonsubstitution method

$$\therefore \int \sin^3 x \cos x\, dx = \frac{\sin^4 x}{4} + C \leftarrow \text{Since form is } \int \left[f(x) \right]^3 f'(x)$$

Example: Evaluate $\int \left(x^2 + 5\right)^{100} x\, dx$

(a) substitution method

$Let\ u = x^2 + 5 \quad then\ du = 2x\, dx$

$$\int \left(x^2 + 5\right)^{100} x\, dx = \frac{1}{2} \int \left(x^2 + 5\right)^{100} \left(2x\, dx\right)$$

$$= \frac{1}{2} \int u^{100}\, du = \frac{1}{2} \cdot \frac{u^{101}}{101} + C = \frac{\left(x^2 + 5\right)^{101}}{202} + C$$

(b) nonsubstitution method

$$\int \left(x^2 + 5\right)^{100} x\, dx = \frac{1}{2} \int \left(x^2 + 5\right)^{100} \left(2x\, dx\right)$$

$$= \frac{1}{2} \cdot \frac{\left(x^2 + 5\right)^{101}}{101} + C \leftarrow \text{Since } x^2 + 5 \text{ is the function and } 2x \text{ its derivative.}$$

Example: Find $\int \frac{3x^2 + 1}{x^3 + x - 1} dx$

Solution:

(a) substitution method.

$Let\ u = x^3 + x - 1\ then\ du = 3x^2 + 1 dx$

$$\therefore \int \frac{3x^2 + 1}{x^3 + x - 1} dx = \int \frac{du}{u} = \ln u + C = \ln\left(x^3 + x - 1\right) + C$$

(b) nonsubstitution method

$$\therefore \int \frac{3x^2+1}{x^3+x-1}\,dx = \ln\left(x^3+x-1\right)+C \;\leftarrow Form \int \frac{f'(x)}{f(x)}$$

In these type of questions you can use the substitution method or the nonsubstitution method, whichever appeals to you.

Exercise:
Indicate which one of the two integrals requires substitution and which one can be found without resorting to substitution.

1. (a) $\int x\sqrt{3x-4}\,dx$ (b) $\int x\sqrt{3x^2-4}\,dx$

2. (a) $\int x\left(x^2+5\right)^6 dx$ (b) $\int x\left(x+5\right)^6 dx$

3. (a) $\int \frac{x}{\sqrt{x-1}}\,dx$ (b) $\int \frac{x}{\sqrt{x^2-1}}\,dx$

4. (a) $\int \frac{x^2}{x^3+5}\,dx$ (b) $\int (x-2)^7 (x+3)^2\,dx$

Exercise:
Find the following integrals, using substitution only when necessary.

1. $\int x\sqrt{2x^2+1}\,dx$
2. $\int \frac{3x^2-1}{\left(x^3-x+4\right)^3}\,dx$
3. $\int 2x\sqrt{2x-1}\,dx$
4. $\int \sin x\sqrt{\cos x}\,dx$
5. $\int \tan^3 x\sec^2 x\,dx$
6. $\int 2x\left(4x^2-1\right)^3 dx$
7. $\int \frac{x}{\sqrt{2x^2-5}}\,dx$
8. $\int \frac{3x}{\left(4-x^2\right)}\,dx$
9. $\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx$

Definite integrals with substitution

$$\int_1^3 (x-1)^2\,dx = \left.\frac{(x-1)^3}{3}\right|_1^3 = \frac{8}{3} - 0 = \frac{8}{3}$$

"graph"

***If substitution is used we MUST change the limits at the APPROPRIATE time.

$Let\ u = x-1 \quad then\ du = dx\ and\ when$

$\underline{x}$	$\underline{u}$
1	0
3	2

$$\int_1^3 (x-1)^2\,dx$$

$$= \int_0^2 u^2\,du = \left.\frac{u^3}{3}\right|_0^2 = \frac{8}{3} - 0 = \frac{8}{3}$$

"graph"

N.B. The limits must be changed!!!

Alternate notation

$Let\ u = x-1 \quad then\ du = dx$

$$\int_1^3 (x-1)^2\,dx$$

$$= \int_{x=1}^{x=3} u^2\,du = \left.\frac{u^3}{3}\right|_{x=1}^{x=3} = \left.\frac{(x-1)^3}{3}\right|_1^3 = \frac{8}{3} - 0 = \frac{8}{3}$$

Examples: Evaluate each definite integral.

$(a)\quad \int_0^{\frac{1}{2}} \frac{x}{\sqrt{1-x^2}}\,dx \qquad (b)\quad \int_0^4 2x\sqrt{4-x}\,dx$

$(c)\quad \int_{-1}^0 x\left(x^2-1\right)^4 dx \qquad (d)\quad \int_{-1}^2 (x+1)(2-x)^4\,dx$

Solutions:

(a) $$\int_0^{\frac{1}{2}} \frac{\left(1-x^2\right)^{-\frac{1}{2}}}{-2}\left(-2x\,dx\right) = -\frac{1}{2}\left(1-x^2\right)^{\frac{1}{2}} 2\Big|_0^{\frac{1}{2}} = -\sqrt{1-x^2}\Big|_0^{\frac{1}{2}} = -\sqrt{\frac{3}{4}}+\sqrt{1}$$

(b) Requires substitution.

Let $u = 4 - x$ *then* $du = -dx$ *and*

x	u
0	4
4	0

$$\int_0^4 2x\sqrt{4-x}\,dx = -\int_4^0 2\left(4-u\right)u^{\frac{1}{2}}\,du$$

$$= -2\int_4^0 \left(4u^{\frac{1}{2}} - u^{\frac{3}{2}}\right)du$$

$$= -2\left[\frac{8}{3}u^{\frac{3}{2}} - \frac{2}{5}u^{\frac{5}{2}}\right]_4^0 = 0 + \left[\frac{16}{3}(4)^{\frac{3}{2}} - \frac{4}{5}(4)^{\frac{5}{2}}\right]$$

$$= \frac{16}{3}(8) - \frac{4}{5}(32)$$

Try (c) and (d) for homework.