Trig Substitution

The types we encounter are: Diagram

$\sqrt{1-\square^2}$ *common form* $\frac{1}{\sqrt{1-\square^2}}$ use $Let\ \square=\sin\theta$

$1+\square^2$ *common form* $\frac{1}{1+\square^2}$ use $Let\ \square=\tan\theta$

$\sqrt{\square^2-1}$ *common form* $\frac{1}{\sqrt{\square^2-1}}$ use $Let\ \square=\sec\theta$

Sine

Basic: Find $\int\frac{1}{\sqrt{1-x^2}}dx$

$$Let\ x=\sin\theta \quad then\ dx=\cos\theta d\theta$$

$$\int\frac{1}{\sqrt{1-x^2}}dx=\int\frac{\cos\theta d\theta}{\sqrt{1-\sin^2\theta}}=\int\frac{\cos\theta d\theta}{\cos\theta}=\int d\theta=\theta+C=\sin^{-1}x+C$$

Patterns:

$$(a)\int\frac{x}{\sqrt{1-x^2}}dx=\int\sin\theta\frac{\cos\theta\, d\theta}{\sqrt{1-\sin^2\theta}}$$

$Let\ x=\sin\theta$

$dx=\cos\theta d\theta$

$$=\int\sin\theta\left(\frac{\cos\theta d\theta}{\cos\theta}\right)=\int\sin\theta\, d\theta$$

"sketch"

$$=-\cos\theta+C=-\sqrt{1-x^2}+C$$

$$(b)\int\frac{x^2}{\sqrt{1-x^2}}dx=\int\sin^2\theta\left(\frac{\cos\theta d\theta}{\cos\theta}\right)=\int\sin^2\theta\, d\theta$$

$Let\ x=\sin\theta$

$dx=\cos\theta d\theta$

$$=\int\frac{1-\cos2\theta}{2}d\theta=\frac{1}{2}\theta-\frac{\sin2\theta}{4}+C$$

$$=\frac{1}{2}\sin^{-1}x-\frac{2\sin\theta\cos\theta}{4}+C=\frac{1}{2}\sin^{-1}x-\frac{1}{2}x\sqrt{1-x^2}+C$$

{Notice how the integrals of powers of the trig functions come into these solutions.}

$$(c)\int \frac{x^3}{\sqrt{1-x^2}}\,dx = \int \sin^3\theta\left(\frac{\cos\theta d\theta}{\cos\theta}\right) = \int \sin^3\theta\,d\theta \qquad Let\ x=\sin\theta,\ dx=\cos\theta d\theta$$

$$= \int \sin^2\theta\sin\theta\,d\theta = \int\left(1-\cos^2\theta\right)\sin\theta\,d\theta$$

$$= \int \sin\theta\,d\theta - \int \cos^2\theta\left(\sin\theta\,d\theta\right) = -\cos\theta + \frac{\cos^3\theta}{3} + C$$

$$= -\sqrt{1-x^2} + \frac{\left(1-x^2\right)\sqrt{1-x^2}}{3} + C$$

Let's place more x's into the denominator.

$$(d)\int \frac{1}{x\sqrt{1-x^2}}\,dx = \int \frac{\cos\theta d\theta}{\sin\theta\sqrt{1-\sin^2\theta}} = \int \frac{d\theta}{\sin\theta} = \int \csc\theta\,d\theta \qquad Let\ x=\sin\theta,\ dx=\cos\theta d\theta$$

$$= \ln\left|\csc\theta-\cot\theta\right| + C = \ln\left|\frac{1}{x}-\frac{\sqrt{1-x^2}}{x}\right| + C$$

$$(e)\int \frac{1}{x^2\sqrt{1-x^2}}\,dx = \int \frac{\cos\theta d\theta}{\sin^2\theta\sqrt{1-\sin^2\theta}} = \int \frac{\cos\theta d\theta}{\sin^2\theta\cos\theta} \qquad Let\ \ x=\sin\theta,\ dx=\cos\theta d\theta$$

$$= \int \frac{d\theta}{\sin^2\theta} = \int \csc^2\theta\,d\theta = -\cot\theta + C = -\frac{\sqrt{1-x^2}}{x} + C$$

$$(f)\int \frac{1}{x^3\sqrt{1-x^2}}\,dx = \int \frac{\cos\theta d\theta}{\sin^3\theta\sqrt{1-\sin^2\theta}} = \int \frac{\cos\theta d\theta}{\sin^3\theta\cos\theta} \qquad Let\ \ x=\sin\theta,\ dx=\cos\theta d\theta$$

$$= \int \frac{d\theta}{\sin^3\theta} = \int \csc^3\theta\,d\theta \leftarrow \text{difficult, done by parts}$$

$$(g)\int \frac{1}{x^4\sqrt{1-x^2}}\,dx = \int \frac{\cos\theta d\theta}{\sin^4\theta\sqrt{1-\sin^2\theta}} = \int \frac{\cos\theta d\theta}{\sin^4\theta\cos\theta} \qquad Let\ \ x=\sin\theta,\ dx=\cos\theta d\theta$$

$$= \int \frac{d\theta}{\sin^4\theta} = \int \csc^4\theta\,d\theta = \int \csc^2\theta\left(\csc^2\theta\,d\theta\right)$$

$$= \int\left(\cot^2\theta+1\right)\left(\csc^2\theta\,d\theta\right) = \int \cot^2\theta\csc^2\theta\,d\theta + \int \csc^2\theta\,d\theta$$

$$= -\frac{\cot^3\theta}{3} - \cot\theta + C = -\frac{1}{3}\left(\frac{\sqrt{1-x^2}}{x}\right)^3 - \frac{\sqrt{1-x^2}}{x} + C$$

Tangent

Basic: Find $\int \frac{1}{1+x^2}dx$

$$Let\ \ x=\tan\theta \quad then\ \ dx=\sec^2\theta d\theta$$

$$\int \frac{1}{1+x^2}dx=\int \frac{\sec^2\theta d\theta}{1+\tan^2\theta}=\int \frac{\sec^2\theta d\theta}{\sec^2\theta}=\int d\theta=\theta+C=\tan^{-1}x+C$$

Patterns:

$(a)\int \frac{x}{1+x^2}dx\ =\int \frac{\tan\theta}{1+\tan^2\theta}\sec^2\theta\, d\theta$ $\qquad$ $Let\ x=\tan\theta$

“sketch”

$=\int \tan\theta\, d\theta=\ln\left|\sec\theta\right|+C=\ln\left|\sqrt{1+x^2}\right|+C$ $\qquad$ $dx=\sec^2\theta d\theta$

[Note: We can also find this integral directly since $\left(1+x^2\right)'=2x$. Therefore,

$$\int \frac{x}{1+x^2}dx=\frac{1}{2}\int \frac{2x}{1+x^2}dx=\frac{1}{2}\ln\left|1+x^2\right|+C$$ ←Same answer as above.]

$(b)\int \frac{x^2}{1+x^2}dx\ =\int \frac{\tan^2\theta}{1+\tan^2\theta}\sec^2\theta\, d\theta$ $\qquad$ $Let\ x=\tan\theta$

$=\int \tan^2\theta\, d\theta=\int\left(\sec^2\theta-1\right)d\theta=\tan\theta-\theta+C$ $\qquad$ $dx=\sec^2\theta d\theta$

$=x-\tan^{-1}x+C$

[Note: We can also do this question more directly by writing $\frac{x^2}{1+x^2}$ as $1-\frac{1}{1+x^2}$

and $\int 1-\frac{1}{1+x^2}dx=x-\tan^{-1}x+C$]

Placing more x’s in the denominator.

$(c)\int \frac{1}{x\left(1+x^2\right)}dx\ =\int \frac{1}{\tan\theta\left(1+\tan^2\theta\right)}\sec^2\theta\, d\theta$ $\qquad$ $Let\ x=\tan\theta$

$=\int \frac{1}{\tan\theta}d\theta=\int \cot\theta\, d\theta=\int \frac{\cos\theta}{\sin\theta}d\theta=\ln\left|\sin\theta\right|+C$ $\qquad$ $dx=\sec^2\theta d\theta$

$=\ln\left|\frac{x}{\sqrt{1+x^2}}\right|+C$

$$(d)\int \frac{1}{x^2\left(1+x^2\right)}dx = \int \frac{1}{\tan^2\theta\left(1+\tan^2\theta\right)}\sec^2\theta\, d\theta \qquad Let\ x=\tan\theta$$

$$= \int \frac{1}{\tan^2\theta}d\theta = \int \cot^2\theta\, d\theta = \int\left(\csc^2\theta - 1\right)d\theta \qquad dx = \sec^2\theta d\theta$$

$$= -\cot\theta - \theta + C = -\left(\frac{1}{x}\right) - \tan^{-1}x + C$$

<u>Secant</u>

<u>"Basic"</u>: Find $\int \frac{1}{\sqrt{x^2-1}}dx$

$$Let\ \ x=\sec\theta \quad then\ \ dx = \sec\theta\tan\theta d\theta$$

$$\int \frac{1}{\sqrt{x^2-1}}dx = \int \frac{\sec\theta\tan\theta d\theta}{\sqrt{\sec^2\theta - 1}} = \int \frac{\sec\theta\tan\theta d\theta}{\tan\theta}$$

"<u>sketch</u>"

$$= \int \sec\theta d\theta = \ln\left|\sec\theta + \tan\theta\right| + C = \ln\left|x+\sqrt{x^2-1}\right| + C$$

This is not really the "Basic" since the answer is not $\sec^{-1}x$. The next integral should be the "Basic".

$$(a)\int \frac{dx}{x\sqrt{x^2-1}} \qquad Let\ \ x=\sec\theta \quad then\ \ dx=\sec\theta\tan\theta d\theta$$

$$\therefore \int \frac{1}{x\sqrt{x^2-1}}dx = \int \frac{\sec\theta\tan\theta d\theta}{\sec\theta\sqrt{\sec^2\theta-1}} = \int \frac{\sec\theta\tan\theta d\theta}{\sec\theta\tan\theta}$$

$$= \int d\theta = \theta + C = \sec^{-1}x + C$$

$$(b)\int \frac{1}{x^2\sqrt{x^2-1}}dx = \int \frac{\sec\theta\tan\theta d\theta}{\sec^2\theta\sqrt{\sec^2\theta-1}} = \int \frac{\sec\theta\tan\theta d\theta}{\sec^2\theta\tan\theta} \qquad Let\ x=\sec\theta \quad dx=\sec\theta\tan\theta d\theta$$

$$= \int \frac{1}{\sec\theta}d\theta = \int \cos\theta\, d\theta = \sin\theta + C = \frac{\sqrt{x^2-1}}{x} + C$$

$$(c)\ \int \frac{1}{x^3\sqrt{x^2-1}}dx = \int \frac{\sec\theta\tan\theta d\theta}{\sec^3\theta\sqrt{\sec^2\theta-1}} = \int \frac{\sec\theta\tan\theta d\theta}{\sec^3\theta\tan\theta} \qquad \begin{aligned} &Let\ x=\sec\theta \\ &dx = \sec\theta\tan\theta d\theta \end{aligned}$$

$$= \int \frac{1}{\sec^2\theta}d\theta = \int \cos^2\theta\, d\theta = \int \frac{1+\cos 2\theta}{2}d\theta$$

$$= \frac{1}{2}\theta + \frac{\sin 2\theta}{4} + C = \frac{1}{2}\theta + \frac{2\sin\theta\cos\theta}{4} + C = \frac{1}{2}\sec^{-1}x + \frac{1}{2}\frac{\sqrt{x^2-1}}{\sqrt{x}}\cdot\frac{1}{x} + C$$

Placing more x's in the numerator.

$$(d)\ \int \frac{x}{\sqrt{x^2-1}}dx = \int \frac{(\sec\theta)\sec\theta\tan\theta d\theta}{\sqrt{\sec^2\theta-1}} = \int \frac{\sec^2\theta\tan\theta d\theta}{\tan\theta} \qquad \begin{aligned} &Let\ x=\sec\theta \\ &dx = \sec\theta\tan\theta d\theta \end{aligned}$$

$$= \int \sec^2 d\theta = \tan\theta + C = \sqrt{x^2-1} + C$$

[Note: You can also find this integral directly since $\left(x^2-1\right)' = 2x$.]

$$(e)\ \int \frac{x^2}{\sqrt{x^2-1}}dx = \int \frac{(\sec^3\theta)\sec\theta\tan\theta d\theta}{\sqrt{\sec^2\theta-1}} = \int \frac{\sec^3\theta\tan\theta d\theta}{\tan\theta} \qquad \begin{aligned} &Let\ x=\sec\theta \\ &dx = \sec\theta\tan\theta d\theta \end{aligned}$$

$$= \int \sec^3 d\theta \qquad \leftarrow \text{difficult one by parts}$$

Generalizing:

1. Let's change the number 1.

$$(a)\qquad \int \frac{1}{\sqrt{4-x^2}}dx \qquad Let\ \ x = 2\sin\theta \qquad then\ \ dx = 2\cos\theta d\theta \qquad \text{"}sketch\text{"}$$

$$\int \frac{1}{\sqrt{4-x^2}}dx = \int \frac{2\cos\theta d\theta}{\sqrt{4-4\sin^2\theta}} = \int \frac{2\cos\theta d\theta}{2\sqrt{1-\sin^2\theta}} = \int d\theta = \theta + C = \sin^{-1}\frac{x}{2} + C$$

$\therefore$ In general $\displaystyle\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + C$

(b) $\int \frac{1}{9+x^2}dx$ *Let* $x = 3\tan\theta$ *then* $dx = 3\sec^2\theta d\theta$

$$\int \frac{1}{9+x^2}dx = \int \frac{3\sec^2\theta d\theta}{9+9\tan^2 x} = \int \frac{3\sec^2\theta d\theta}{9\left(1+\tan^2\theta\right)} = \frac{1}{3}\int d\theta$$ "sketch"

$$= \frac{1}{3}\theta + C = \frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right) + C$$

$\therefore$ In general $\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$

(c) $\int \frac{dx}{x\sqrt{x^2-25}}$ *Let* $x = 5\sec\theta$ *then* $dx = 5\sec\theta\tan\theta d\theta$

$$= \int \frac{5\sec\theta\tan\theta d\theta}{5\sec\theta\sqrt{25\sec^2\theta - 25}} = \int \frac{\tan\theta d\theta}{5\sqrt{\sec^2\theta - 1}}$$ "*sketch*"

$$= \int \frac{d\theta}{5} = \frac{1}{5}\theta + C = \frac{1}{5}\sec^{-1}\left(\frac{x}{5}\right) + C$$

$\therefore$ In general $\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}\frac{x}{a} + C$

(d) $\int \frac{dx}{4+9x^2}$ *Let* $3x = 2\tan\theta$ *then* $dx = \frac{2}{3}\sec^2\theta\, d\theta$

"sketch"

$$= \int \frac{\frac{2}{3}\sec^2\theta d\theta}{4+4\tan^2\theta} = \int \frac{\frac{2}{3}\sec^2\theta d\theta}{4\left(1+\tan^2\theta\right)} = \frac{1}{6}\int d\theta = \frac{1}{6}\tan^{-1}\left(\frac{3x}{2}\right) + C$$

$$(e)\qquad \int \frac{dx}{x\sqrt{4-25x^2}} \qquad \text{Let } 5x = 2\sin\theta \quad \text{then} \quad dx = \frac{2}{5}\cos\theta d\theta$$

$$= \int \frac{\frac{2}{5}\cos\theta d\theta}{\frac{2}{5}\sin\theta\sqrt{4-4\sin^2\theta}} = \int \frac{\cos\theta d\theta}{\sin\theta(2)\sqrt{1-\sin^2\theta}} \qquad \text{"sketch"}$$

$$= \frac{1}{2}\int \csc\theta\, d\theta = \frac{1}{2}\ln\left|\csc\theta - \cot\theta\right| + C = \frac{1}{2}\ln\left|\frac{2}{5x} - \frac{\sqrt{4-25x^2}}{5x}\right| + C$$

Fractional exponents:

$$(f)\qquad \int \frac{dx}{\left(4+x^2\right)^{\frac{3}{2}}} \qquad \text{Let } x = 2\tan\theta \qquad \text{then} \qquad dx = 2\sec^2\theta d\theta$$

$$= \int \frac{2\sec^2\theta d\theta}{\left(4+4\tan^2\theta\right)^{\frac{3}{2}}} = \int \frac{2\sec^2\theta d\theta}{4^{\frac{3}{2}}\left(1+\tan^2\theta\right)^{\frac{3}{2}}} = \int \frac{2\sec^2\theta d\theta}{8\left(\sec^2\theta\right)^{\frac{3}{2}}} \qquad \text{"sketch"}$$

$$= \frac{1}{4}\int \frac{\sec^2\theta d\theta}{\sec^3\theta} = \frac{1}{4}\int \frac{d\theta}{\sec\theta} = \frac{1}{4}\int \cos\theta\, d\theta$$

$$= \frac{1}{4}\sin\theta + C = \frac{1}{4}\frac{x}{\sqrt{x^2+4}} + C$$

$$(g)\qquad \int \frac{dx}{\left(9x^2-1\right)^{\frac{3}{2}}} \qquad \text{Let } 3x = \sec\theta \qquad \text{then} \qquad 3dx = \sec\theta\tan\theta d\theta$$

$$= \int \frac{\sec\theta\tan\theta d\theta}{3\left(\sqrt{\sec^2\theta - 1}\right)^3} = \int \frac{\sec\theta\tan\theta d\theta}{3\tan^3\theta} = \frac{1}{3}\int \frac{\sec\theta d\theta}{\tan^2\theta} \qquad \text{"sketch"}$$

$$= \frac{1}{3}\int \frac{\frac{1}{\cos\theta}d\theta}{\frac{\sin^2\theta}{\cos^2\theta}} = \frac{1}{3}\int \frac{\cos\theta d\theta}{\sin^2\theta} = \frac{1}{3}\int \left(\sin\theta\right)^{-2}\cos\theta\, d\theta$$

$$= \frac{1}{3}\frac{\left(\sin\theta\right)^{-1}}{-1} + C = -\frac{1}{3}\csc\theta + C = -\frac{1}{3}\frac{3x}{\sqrt{9x^2-1}} + C$$

2. <u>Completing the Square</u>

$$\int \frac{1}{x^2-2x+5}dx = \int \frac{1}{\left(x^2-2x+1-1\right)+5}dx = \int \frac{1}{\left(x-1\right)^2+4}dx \leftarrow \text{completed square}$$

$$\textit{Let} \quad x-1 = 2\tan\theta \qquad \textit{then} \;\; dx = 2\sec^2\theta d\theta$$

$$\therefore \int \frac{1}{\left(x-1\right)^2+4}dx = \int \frac{2\sec^2\theta d\theta}{4\tan^2\theta+4} = \int \frac{2\sec^2\theta d\theta}{4\left(\tan^2\theta+1\right)} = \frac{1}{2}\int d\theta \qquad \text{"}\textit{sketch}\text{"}$$

$$= \frac{1}{2}\theta + C = \frac{1}{2}\tan^{-1}\left(\frac{x-1}{2}\right)+C$$

$$(b) \qquad \int \frac{1}{\sqrt{4-2x-x^2}}dx = \int \frac{1}{\sqrt{4-\left(x^2+2x+1-1\right)}}dx = \int \frac{1}{\sqrt{5-\left(x+1\right)^2}}dx$$

$$\textit{Let} \quad x+1 = \sqrt{5}\sin\theta \;\; \textit{then} \;\; dx = \sqrt{5}\cos\theta d\theta$$

$$\therefore \int \frac{1}{\sqrt{5-\left(x+1\right)^2}}dx = \int \frac{\sqrt{5}\cos\theta d\theta}{\sqrt{5-5\sin^2\theta}} = \int \frac{\sqrt{5}\cos\theta d\theta}{\sqrt{5}\sqrt{1-\sin^2\theta}} \qquad \text{"}\textit{sketch}\text{"}$$

$$= \int d\theta = \theta + C = \sin^{-1}\left(\frac{x+1}{\sqrt{5}}\right)+C$$

$$(c) \qquad \int \frac{1}{\sqrt{5-4x-2x^2}}dx = \int \frac{1}{\sqrt{5-2\left(x^2+2x+1-1\right)}}dx = \int \frac{1}{\sqrt{7-2\left(x+1\right)^2}}dx$$

$$\textit{Let} \quad \sqrt{2}\left(x+1\right) = \sqrt{7}\sin\theta \;\; \textit{then} \;\; \sqrt{2}dx = \sqrt{7}\cos\theta d\theta$$

$$\therefore \int \frac{\frac{\sqrt{7}}{\sqrt{2}}\cos\theta}{\sqrt{7-7\sin^2\theta}}d\theta = \frac{\sqrt{7}}{\sqrt{2}}\int \frac{\cos\theta d\theta}{\sqrt{7}\sqrt{1-\sin^2\theta}} = \frac{1}{\sqrt{2}}\int \frac{\cos\theta d\theta}{\cos\theta} \qquad \text{"}\textit{sketch}\text{"}$$

$$= \frac{1}{\sqrt{2}}\int d\theta = \frac{1}{\sqrt{2}}\theta + C = \frac{1}{\sqrt{2}}\sin^{-1}\left(\frac{\sqrt{2}\left(x+1\right)}{\sqrt{7}}\right)+C$$

Variations and generalizations.

If $\sqrt{1-x^2}$ is in the numerator of this famous integral which finds the area of a circle with radius 1.

Example: Find the area of a circle with radius = 1. "sketch"

Solution:

$$4\int_0^1 \sqrt{1-x^2}\,dx \qquad \text{Let } x=\sin\theta \quad \text{then} \quad dx=\cos\theta d\theta$$

x	θ
0	0
1	$\frac{\pi}{2}$

$$=4\int_0^{\frac{\pi}{2}} \sqrt{1-\sin^2\theta}\cos\theta d\theta = 4\int_0^{\frac{\pi}{2}} \cos^2\theta\, d\theta = 4\int_0^{\frac{\pi}{2}} \frac{1+\cos 2\theta}{2} d\theta$$

$$=4\left(\frac{1}{2}\theta+\frac{\sin 2\theta}{4}\Big|_0^{\frac{\pi}{2}}\right) = 4\cdot\frac{1}{2}\left(\frac{\pi}{2}\right)+0-[0] = 4\left(\frac{\pi}{4}\right)=\pi$$

Let's look at this problem using parametric equations and using a polar curve.

Parametric

A circle with radius=1 is defined by parametric equations:

$$x=\cos\theta \qquad y=\sin\theta$$

$$\therefore \text{Area } = -\int_0^{2\pi} y\frac{dx}{d\theta}d\theta = -\int_0^{2\pi}(\sin\theta)(-\sin\theta)d\theta$$

$$=\int_0^{2\pi}\sin^2\theta\, d\theta = \int_0^{2\pi}\frac{1-\cos 2\theta}{2}d\theta$$

$$=\frac{1}{2}\theta-\frac{\sin 2\theta}{4}\Big|_0^{2\pi} \qquad \leftarrow \text{Basically the same}$$

$$=\frac{1}{2}(2\pi)-0-[0]=\pi$$

Polar

A circle with radius = 1 has equation r = 1.

$$\therefore Area = \frac{1}{2}\int_0^{2\pi} r^2\, d\theta = \frac{1}{2}\int_0^{2\pi} 1^2 d\theta = \frac{1}{2}\theta\Big|_0^{2\pi} = \frac{1}{2}(2\pi)-0=\pi$$

Which system is more efficient? Do you see the advantages of different systems?