Trigonometric Integrals

The easiest integral is $\int f'(x)\,dx$ since the answer is simply $f(x)$.

Hence, the following six trigonometric integral, other than a possible sign change, are easy:

Type 1

$\int \cos x\,dx$	$\int \sin x\,dx$	$\int \sec^2 x\,dx$
$\int \csc^2 x\,dx$	$\int \sec x \tan x\,dx$	$\int \csc x \cot x\,dx$

Furthermore, the integral $\int \left(f(x)\right)^n f'(x)\,dx = \dfrac{\left(f(x)\right)^{n+1}}{n+1} + C \;\; with\, n \neq -1$ is also easy.

Type 2

Hence,

$\int \sin^5 x \cos x\,dx =$

$\int \cos^4 x \sin x\,dx =$

$\int \tan^2 x \sec^2 x\,dx =$

$\int \cot^3 x \csc^2 x\,dx =$

$\int \sec^5 x \tan x\,dx =$

$\int \csc^6 x \cot x\,dx =$

are all easy integrals.

Sometimes we must use identities to change the integrals into easy integrals.

For example,

$\int \sin^3 x\,dx \leftarrow$ the function $\sin x$ is there, but we are missing its derivative!

Using identities we produce the derivative as follows:

$$\int \sin^3 x\,dx = \int \sin^2 x \sin x\,dx = \int \left(1-\cos^2 x\right)\sin x\,dx$$

$$= \int \sin x\,dx - \int \cos^2 x \sin x\,dx = -\cos x + \frac{\cos^3 x}{3} + C$$

The identity changed the question into one type1 and one type 2 integral.

Example: Find $\int \sec^4 x\,dx$

Solution:

$\sec^4 x$ is not a derivative. However, $\sec^2 x$ is a derivative of $\tan x$.

$$\int \sec^4 x\,dx = \int \sec^2 x \sec^2 x\,dx = \int \left(\tan^2 x + 1\right)\sec^2 x\,dx$$

$$= \int \left(\tan^2 x\right)\sec^2 x\,dx + \int \sec^2 x\,dx = \frac{\tan^3 x}{3} + \tan x + C$$

Again, the identity changed the question into one type1 and one type 2 integral.

Sometimes we must resort to the double angle identities.

Example: Find $\int \sin^2 x\,dx$

Solution:

If we write

$\sin^2 x = \sin x \cdot \sin x$ it will not do us much good since $\sin x$ is not the derivativ of $\sin x$.

We will use the identity $\cos 2x = 1 - 2\sin^2 x$ in the form $\sin^2 x = \frac{1-\cos 2x}{2}$, which removes the exponent in the problem.

Therefore,

$$\int \sin^2 x\,dx = \int \frac{1-\cos 2x}{2}\,dx = \int \frac{1}{2}\,dx - \int \frac{\cos 2x}{2}\,dx = \frac{1}{2}x - \frac{\sin 2x}{4} + C$$

Contrast this solution with the solution of $\int \sin^3 x\,dx$.

We have looked at finding the integrals of the first three powers of $\sin x$. It will be beneficial to find the integrals of the first three powers of the other five trigonometric functions. Try these yourself and then check the answers.

Cosine Find:

$(a)\quad \int \cos x\,dx \qquad (b)\quad \int \cos^2 x\,dx \qquad (c)\quad \int \cos^3 x\,dx$

Solutions: Very similar to integrals of powers of the $\sin x$ function.

$(a)\quad \int \cos x\,dx = \sin x + C$

(b) $\int \cos^2 x\,dx = \int \frac{1+\cos 2x}{2}dx = \int \frac{1}{2}dx + \int \frac{\cos 2x}{2}dx = \frac{1}{2}x + \frac{\sin 2x}{4} + C$

(c) $\int \cos^3 x\,dx = \int \cos^2 x \cos x\,dx = \int \left(1-\sin^2 x\right)\cos x\,dx$

$$= \int \cos x\,dx - \int \sin^2 x \cos x\,dx = \sin x - \frac{\sin^3 x}{3} + C$$

<u>Tangent</u> Find:

(a) $\int \tan x\,dx$ (b) $\int \tan^2 x\,dx$ (c) $\int \tan^3 x\,dx$

Solutions:

We keep looking for the derivative $f'(x)$!!

(a) $\int \tan x\,dx = \int \frac{\sin x}{\cos x}dx = -\ln\left|\cos x\right| + C \text{ or } \ln\left|\sec x\right| + C \leftarrow \text{since } \cos x = \frac{1}{\sec x}$

(b) $\int \tan^2 x\,dx = \int \left(\sec^2 x - 1\right)dx = \tan x - x + C$

(c) $\int \tan^3 x\,dx = \int \tan x \tan^2 x\,dx = \int \tan x\left(\sec^2 x - 1\right)dx$

$$= \int \tan x\left(\sec^2 x\,dx\right) - \int \tan x\,dx = \frac{\tan^2 x}{2} + \ln\left|\cos x\right| + C$$

<u>Cotangent</u> Find:

(a) $\int \cot x\,dx$ (b) $\int \cot^2 x\,dx$ (c) $\int \cot^3 x\,dx$

Solutions: Very similar to integrals of powers of the $\tan x$ function.

(a) $\int \cot x\,dx = \int \frac{\cos x}{\sin x}dx = \ln\left|\sin x\right| + C$

(b) $\int \cot^2 x\,dx = \int \left(\csc^2 x - 1\right)dx = -\cot x - x + C$

(c) $\int \cot^3 x\,dx = \int \cot x \cot^2 x\,dx = \int \cot x\left(\csc^2 x - 1\right)dx$

$$= \int \cot x\left(\csc^2 x\,dx\right) - \int \cot x\,dx = -\frac{\cot^2 x}{2} - \ln\left|\sin x\right| + C$$

<u>Secant</u> Find:

(a) $\int \sec x\,dx$ (b) $\int \sec^2 x\,dx$ (c) $\int \sec^3 x\,dx$

Solutions:

sec$x$ is not a derivative of our know functions. However, sec$x$tan$x$ and sec^2x are derivatives we know. The method is to form both of these known derivatives.

(a) $\int \sec x\,dx = \int \sec x\left(\dfrac{\tan x + \sec x}{\tan x + \sec x}\right)dx = \int \dfrac{\sec x \tan x + \sec^2 x}{\tan x + \sec x}dx$

$$= \ln\left|\tan x + \sec x\right| + C \leftarrow \text{since we created } \int \frac{f'(x)}{f(x)}dx$$

(b) $\int \sec^2 x\,dx = \tan x + C$ ←This was easy since $\sec^2 x$ is a known derivative.

(c) $\int \sec^3 x\,dx$ ← This is a difficult integral to be done by parts.

Let $u = \sec x$ $\quad dv = \sec^2 x\,dx$

$du = \sec x \tan x\,dx \quad v = \tan x$

$I = \int \sec^3 x\,dx = \sec x \tan x - \int \tan^2 x \sec x\,dx$ ← Is this new integral easier?

Not really but we can partition it to form the original integral I.

$$I = \sec x \tan x - \int \left(\sec^2 x - 1\right)\sec x\,dx = \sec x \tan x - I + \int \sec x\,dx$$

$2I = \sec x \tan x + \ln\left|\sec x + \tan x\right| + C$ ← By (a) above

$$\therefore I = \int \sec^3 x\,dx = \frac{\sec x \tan x + \ln\left|\sec x + \tan x\right|}{2} + K$$

<u>Cosecant</u> Find:

(a) $\int \csc x\,dx$ $\qquad (b)$ $\int \csc^2 x\,dx$ $\qquad (c)$ $\int \csc^3 x\,dx$

The work will be very similar to the integrals of powers of sec*x*, (*b*) will be very easy whereas (*c*) is challenging but not impossible.

(a) $\int \csc x\,dx = \int \csc x\left(\dfrac{\cot x + \csc x}{\cot x + \csc x}\right)dx = \int \dfrac{\csc x \cot x + \csc^2 x}{\cot x + \csc x}dx$

$$= \ln\left|-\cot x - \csc x\right| + C \leftarrow \text{since we created } \int \frac{f'(x)}{f(x)}dx$$

By multiplying by $\left(\dfrac{-\cot x + \csc x}{-\cot x + \csc x}\right)$ we would arrive at an equivalent value

$\int \csc x\,dx = \ln\left|\csc x - \cot x\right| + C$ ← found in most texts

(b) $\int \csc^2 x\,dx = -\cot x + C$ ←This was easy since $\csc^2 x$ is a known derivative.

(c) $\int \csc^3 x\,dx$ ← This is a difficult integral to be done by parts.

$$Let\ u = \csc x \qquad dv = \csc^2 x\,dx$$

$$du = -\csc x \cot x\,dx \quad v = -\cot x$$

$$I = \int \csc^3 x\,dx = -\csc x \cot x - \int \cot^2 x \csc x\,dx$$ ← Is this new integral easier?

Not really but we can partition it to form the original integral I.

$$I = -\csc x \cot x - \int (\csc^2 x - 1)\csc x\,dx = -\csc x \cot x - I + \int \csc x\,dx$$

$$2I = -\csc x \cot x + \ln|\csc x - \cot x| + C$$ ← By (a) above, preferred form

$$\therefore I = \int \csc^3 x\,dx = \frac{-\csc x \cot x + \ln|\csc x - \cot x|}{2} + K$$

It is advisable that you are familiar with all the above work, although part (c) in the last two is quite difficult. The techniques can be used in finding other integrals.

Some combined examples: ←Remember you are always looking for the basic trig derivatives. Some may be hidden in chain rules.

$$\left.\begin{array}{l} \sin\ \textit{and}\ \cos\ \textit{work together}. \\ \tan\ \textit{and}\ \sec\ \textit{work together}. \\ \cot\ \textit{and}\ \csc\ \textit{work together}. \end{array}\right\} \quad \textit{See Gerhard pg.} 102-103$$