<u>Volume</u>

We will look at three methods of calculating volumes.

1. By slicing a cross-section.
2. By disks
3. by shells.

<u>By slicing a cross-section</u>

$$Area = lw \qquad \therefore A(x) = \int_a^b f(x)\,dx$$

$$Volume = area \times height \qquad \therefore V(x) = \int_a^b A(x)\,dx$$

Example: Find the volume of a right circular cone.

"sketch"

$$A(a) = \pi a^2$$

In terms of $x: A(x) = \pi\left(\frac{rx}{h}\right)^2$ since $\frac{a}{x} = \frac{r}{h}$ so $a = \frac{rx}{h}$

$$= \left(\frac{\pi r^2}{h^2}\right) x^2$$

$$\therefore V(x) = \int_0^h A(x)\,dx = \int_0^h \frac{\pi r^2}{h^2} x^2\,dx$$

$$= \frac{\pi r^2}{h^2} \frac{x^3}{3}\bigg|_0^h = \frac{\pi r^2}{h^2} \frac{h^3}{3} = \frac{1}{3}\pi r^2 h$$

If you look at the above example you should notice that the problem is solve once we have found the area function $A(x)$.

Therefore, it is useful to know the following <u>area</u> formulae.

Cross-section figure	diagram	formula
square		s^2
semi-circle		$\frac{1}{2}\pi r^2$
equilateral triangle		$\frac{1}{2}a\left(\frac{\sqrt{3}a}{2}\right)$

Others we can look at are a triangle and an isosceles triangle.

If the cross-section has its base perpendicular to the *x*-axis we have a formula involving the length *y* or 2*y*. Therefore, know the following:

figure	length is y	length is $2y$
square	y^2	$(2y)^2 = 4y^2$
semi-circle	$\frac{\pi\left(\frac{y}{2}\right)^2}{2}$	$\frac{\pi y^2}{2}$
equilateral triangle	$\frac{1}{2}y\left(\frac{\sqrt{3}}{2}y\right) = \frac{\sqrt{3}y^2}{4}$	$\frac{1}{2}(2y)\left(\sqrt{3}y\right) = \sqrt{3}y^2$

I like to set up a coordinate system when a volume is requested.

“sketch” “area sketch”

Calculating the area $A(y)=\pi y^2$ But $y=\frac{r}{h}x$

$\therefore$ Area in terms of x: $A(x)=\pi\left(\frac{r}{h}x\right)^2=\frac{\pi r^2}{h^2}x^2$

Using the above to find the volume is easy!

$$V(x)=\int_0^h A(x)\,dx=\int_0^h \frac{\pi r^2}{h^2}x^2\,dx$$

$$=\frac{\pi r^2}{h^2}\frac{x^3}{3}\bigg|_0^h=\frac{\pi r^2}{h^2}\frac{h^3}{3}=\frac{\pi r^2 h}{3}$$

Example: A solid has a circular base of radius = 1. Planar cross section perpendicular to the base are:

(a) squares.

(b) equilateral; triangles

Find the volume of the solid.

Solution:

(a) “sketch” “area sketch”

$A(y)=(2y)(2y)=4y^2$ But $x^2+y^2=1$

In terms of x: $A(x)=4\left(1-x^2\right)$

$$V=\int_{-1}^{1}A(x)\,dx=\int_{-1}^{1}4\left(1-x^2\right)dx$$

or $V=2\int_0^1 4\left(1-x^2\right)dx=8\left[x-\frac{x^3}{3}\right]_0^1=\frac{16}{3}$

(b) "sketch" "area sketch"

$$A(y) = \frac{1}{2}bh = \frac{1}{2}(2y)\left(\sqrt{3}y\right) = \sqrt{3}y^2 \quad \text{But } x^2 + y^2 = 1$$

In terms of x: $A(x) = \sqrt{3}\left(1 - x^2\right)$

$$V = \int_{-1}^{1} A(x)\,dx = \int_{-1}^{1} \sqrt{3}\left(1 - x^2\right)dx$$

or $V = 2\int_0^1 \sqrt{3}\left(1 - x^2\right)dx = 2\sqrt{3}\left[x - \frac{x^3}{3}\right]_0^1 = 2\sqrt{3}\left(\frac{2}{3}\right) = \frac{4}{3}\sqrt{3}$

Example (pg. 454 #56)

The base S is the parabolic region $\left\{(x, y) \middle| x^2 \le y \le 1\right\}$. Cross- sections perpendicular to the y-axis are equilateral triangles. Find the volume of S.

Solution:

"sketch" "area sketch"

$$A(x) = \frac{1}{2}(2x)\sqrt{3}x \qquad \text{But } x^2 = y$$

Area in terms of y: $A(y) = \sqrt{3}y$

$$\therefore \text{Volume} = \int_0^1 A(y)\,dy = \int_0^1 \sqrt{3}y\,dy = \left.\frac{\sqrt{3}y^2}{2}\right|_0^1 = \frac{\sqrt{3}}{2}$$

<u>Disk/Washer Method</u>

What is the volume of a cylinder ("hockey puck")?

"sketch" V = area ×height = $\pi r^2 h$

$$V = \pi(2)^2 3 = 12\pi$$

In general,

"sketch" $V = \pi r^2 x$

"sketch" Rotate $f(x)$, from a to b, about the x-axis

$V = \int_a^b \pi\left[f(x)\right]^2 dx$ ←This is the sum of all the cylinders.

Example: (pg. 452 #2)

Find the volume of the solid obtained by rotating the region bounded by the curves $y = e^x,\ y = 0,\ x = 0,\ x = 1$ about the x-axis

Solution:

$$"Sketch" \qquad V = \int_0^1 \pi y^2\, dx = \int_0^1 \pi \left(e^x\right)^2 dx \qquad \leftarrow "\pi r^2 h"$$

$$= \int_0^1 \pi e^{2x}\, dx = \pi \frac{e^{2x}}{2}\Big|_0^1$$

$$= \frac{\pi e^2}{2} - \frac{\pi e^0}{2} = \frac{\pi}{2}\left(e^2 - 1\right)$$

Example: (pg. 452 #2)

Find the volume of the solid obtained by rotating the region bounded by the curves $y = \sqrt{x-1},\ y = 0,\ x = 2,\ x = 5$ about the x-axis

Solution:

$$"Sketch" \qquad V = \int_2^5 \pi y^2\, dx = \int_2^5 \pi \left(\sqrt{x-1}\right)^2 dx \qquad \leftarrow "\pi r^2 h"$$

$$= \pi \int_2^5 (x-1)\, dx = \pi \left[\frac{x^2}{2} - x\right]\Bigg|_2^5$$

$$= \pi \left(\frac{25}{2} - 5 - \frac{4}{2} + 2\right)$$

Example: Find the formula for the volume of a cone

Solution: “sketch” “coordinatized sketch”

$$y = \frac{r}{h} x$$

$$V = \int_0^h \pi y^2\, dx = \int_0^h \pi \left(\frac{r}{h} x\right)^2 dx = \frac{\pi r^2}{h^2} \int_0^h x^2\, dx$$

$$= \frac{\pi r^2}{h^2} \left[\frac{x^3}{3}\right]_0^h = \frac{\pi r^2}{h^2}\left(\frac{h^3}{3}\right) - 0 = \frac{\pi r^2 h}{3} \quad \leftarrow \text{Same as with cross-sections}$$

Example: (pg. 452 #6)

Find the volume of the solid obtained by rotating the region bounded by the curves $x = y - y^2$ and $x = 0$ about the y-axis

Solution:

$$"Sketch" \qquad V = \int_0^1 \pi x^2 \, dy = \int_0^1 \pi \left(y - y^2 \right)^2 dy \qquad \leftarrow "\pi r^2 h"$$

$$= \pi \int_0^1 \left(y^2 - 2y^3 + y^4 \right) dy = \pi \left[\frac{y^3}{3} - \frac{y^4}{2} + \frac{y^5}{5} \right]\Bigg|_0^1$$

$$= \pi \left(\frac{1}{3} - \frac{1}{2} + \frac{1}{5} \right) = \frac{\pi}{30}$$

<u>Washers</u>:

Find the volume of the solid obtained by rotating the region enclosed by the curves $y = x^2 + 2$ and $y = x + 8$ about the x-axis.

Solution:

$$"Sketch" \qquad V = \int_{-2}^{3} \pi y^2 \, dx - \int_{-2}^{3} \pi Y^2 \, dx \qquad \leftarrow "\pi r^2 h"$$

<u>Intersection:</u>

$$= \int_{-2}^{3} \pi \left(x + 8 \right)^2 dx - \int_{-2}^{3} \pi \left(x^2 + 2 \right)^2 dx$$

$x^2 + 2 = x + 8$

$x^2 - x - 6 = 0$

$(x-3)(x+2) = 0$

$x = 3, -2$

OR

$$V = \int_{-2}^{3} \pi \left[\left(x + 8 \right)^2 - \left(x^2 + 2 \right)^2 \right] dx$$

$$= \int_{-2}^{3} \pi \left[x^2 + 16x + 64 - x^4 - 4x^2 - 4 \right] dx$$

$$= \int_{-2}^{3} \pi \left[-x^4 - 3x^2 + 16x + 60 \right] dx \qquad etc.$$

Example: (pg. 452 #12)

Find the volume of the solid obtained by rotating the region bounded by the curves $y = x^2$ and $y = 4$ about the line $y = 4$.

Solution:

$$"Sketch" \qquad V=\int_{-2}^{2}\pi\left(4-y\right)^2 dx \qquad \leftarrow "\pi r^2 h"$$

<u>Intersection</u> OR

$$x^2=4 \qquad V=2\int_0^2\pi\left(4-y\right)^2 dx=2\int_0^2\pi\left(4-x^2\right)^2 dx$$

$$x=\pm 2 \qquad =2\pi\int_0^2\left(16-8x^2+x^4\right)dx$$

$$=2\pi\left[16x-\frac{8x^3}{3}+\frac{x^5}{2}\right]_0^2=\ldots$$

Example: (pg. 452 #12)

Find the volume of the solid obtained by rotating the region bounded by the curves $y=x$ and $y=\sqrt{x}$ about the line x = 2.

Solution:

$$"Sketch" \qquad V=\int_0^1\pi R^2\,dy-\int_0^1\pi r^2\,dy \qquad \leftarrow "\pi r^2 h"$$

<u>Intersection</u>

$$=\int_0^1\pi\left(2-X\right)^2 dy-\int_0^1\pi\left(2-x\right)^2 dy$$

$$x=\sqrt{x} \qquad =\int_0^1\pi\left(2-y^2\right)^2 dy-\int_0^1\pi\left(2-y\right)^2 dy$$

$$x^2=x \qquad =\pi\int_0^1\left(4-4y^2+y^4\right)-\left(4-4y+y^2\right)dy$$

$$x^2-x=0 \qquad =\pi\int_0^1 y^4-5y^2+4y\,dy$$

$$x\left(x-1\right)=0 \qquad =\pi\left[\frac{y^5}{5}-\frac{5y^3}{3}+2y^2\right]_0^1$$

$$x=0,\ x=1 \qquad =\pi\left[\frac{1}{5}-\frac{5}{3}+2\right]$$

$$y=0,\ y=1 \qquad =\frac{8}{15}\pi$$

Shell Method:
The following example demonstrates the need for the shell method.

Example: Find the volume as the region enclosed by the x-axis and the curve $y = 2x^2 - x^3$ is rotated about the y-axis.

Solution:

"sketch"

To use the disk method we would need to change the function into the form $x = f(y)$. This is difficult.

We find the sum of all rectangles rolled into shells

"sketch".

Volume = area × thickness.

area = surface area of the side = circumference × height

= $(2\pi x)y$ or $(2\pi x)f(x)$

thickness = dx

$\therefore$ volume of each shell $= 2\pi x f(x)\,dx$

Total accummulation of all shells = Volume = $\int_a^b 2\pi x f(x)\,dx$

[Note: Using disk think of $\pi r^2 h$, while using shells think of $2\pi rh$.]

Example: Let's do the introductory question to this section.

Solution:

"*sketch*" $y = 2x^2 - x^3$

$$V = \int_0^2 2\pi x\left(2x^2 - x^3\right)dx \qquad \leftarrow \text{"}2\pi rh\text{"}$$

$$V = 2\pi\int_0^2 \left(2x^3 - x^4\right)dx$$

$$V = 2\pi\left[\frac{1}{2}x^4 - \frac{1}{5}x^5\right]_0^2 = 2\pi\left(8 - \frac{32}{5}\right) = \frac{16}{5}\pi$$

Example: Find the volume of the solid obtained by rotating the region bounded by the curves $y = x^2$ and $y = x$ about the y-axis using:
(a) shells (b) disks

Solution:

(a) $\quad Shells \leftarrow "2\pi rh" \qquad "sketch"$

$$V = \int_0^1 2\pi x\left(x - x^2\right)dx = \int_0^1 2\pi\left(x^2 - x^3\right)dx$$

$$= 2\pi\left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = 2\pi\left[\frac{1}{3} - \frac{1}{4}\right] = 2\pi\left[\frac{1}{12}\right] = \frac{\pi}{6}$$

(a) $\quad Shells \leftarrow "\pi r^2 h" \qquad "sketch"$

$$V = \int_0^1 \pi\left(X^2 - x^2\right)dy = \int_0^1 \pi\left(y - y^2\right)dy$$

$$= \pi\left[\frac{y^2}{2} - \frac{y^3}{3}\right]_0^1 = \pi\left[\frac{1}{2} - \frac{1}{3}\right] = \frac{\pi}{6}$$

Example: Use the shell method to find the volume of a hemi-sphere with radius 1.
Solution:

$$"2\pi rh" \qquad x^2 + y^2 = 1 \quad \therefore x = \sqrt{1 - y^2}$$

"sketch" $\quad V = \int_0^1 2\pi y\sqrt{1 - y^2}\, dy$

$$V = -\pi\left(1 - y^2\right)^{\frac{3}{2}}\left(\frac{2}{3}\right)\Bigg|_0^1 = -\pi\left(0 - \frac{2}{3}\right) = \frac{2}{3}\pi$$

Example: Set up an integral to find the volume of the solid obtained by rotating the region bounded by the curves $y = 4x - x^2$ and $y = x$ about the line with equation $x = 7$.

Solution:

"*sketch*" Shells is much easier $\therefore$ "$2\pi rh$"

<u>Intersection</u>: $V = \int_0^3 2\pi(7-x)\left[\left(4x-x^2\right)-x\right]dx$

$4x - x^2 = x$

$0 = x^2 - 3x$

$0 = x(x-3)$

$x = 0,\ 3$