Chapter 2 (Sequences)

Solve for x:

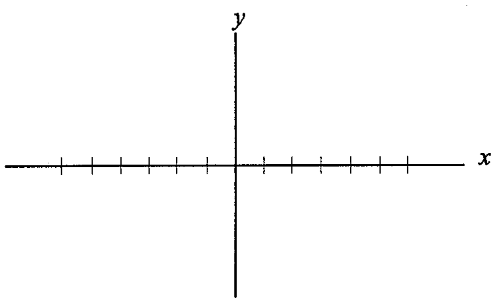
(a) $x^2 - x = 2$

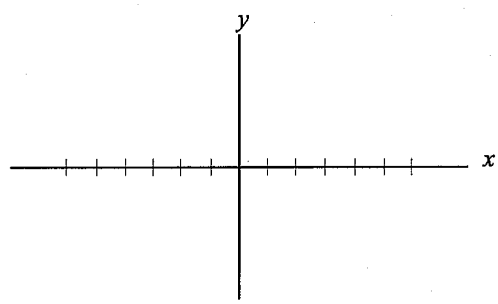
(b) $\sin x = x$

Sketch and state the domain.

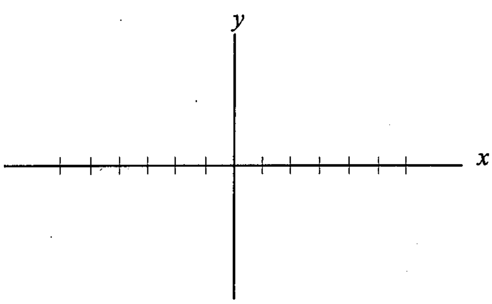
(a) $f(x) = x^2$

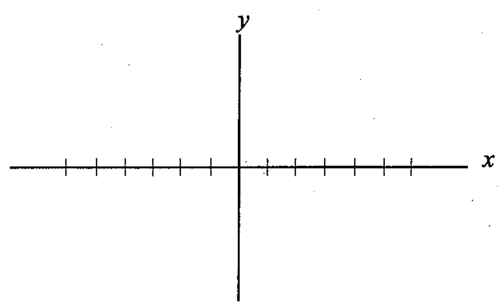
(b) $g(x) = \dfrac{x}{x+1}$





Sketch the above functions if the domains are restricted to the Natural Numbers.





Notation:

(a) $a_n = n^2$

or $\{n^2\}_1^\infty$

or $\{n^2\}$

(b) $b_n = \dfrac{n}{n+1}$

or $\left\{\dfrac{n}{n+1}\right\}_1^\infty$

or $\left\{\dfrac{n}{n+1}\right\}$

Definition:
An infinite sequence is a function whose domain is the set of positive integers.

Note: The sequence of odd positive integers is: 1, 3, 5, … BAD!

Better: $\{2n-1\}$ or $\{2n-1\}_1^\infty$

Examples:
Write the first five terms of each sequence.

(a) $\left\{\dfrac{n^2+2}{n^2+n+2}\right\}$ (b) $\left\{\dfrac{1}{n}\right\}$ (c) $\{3^n\}$

(d) $\{\sqrt{n-3}\}_3^\infty$ (e) $\{\cos n\pi\}$ (f) $\{(-1)^{n+1}\}$

↑

Required if domain ≠ N.

Examples:

Find an explicit n^{th} term definition for each infinite sequence. (Assume the most obvious pattern continues.)

(a) $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots$

(b) $\frac{2}{5}, \frac{4}{8}, \frac{6}{11}, \frac{8}{14}, \ldots$

(c) $\frac{2}{3}, \frac{3}{9}, \frac{4}{27}, \frac{5}{81}, \ldots$

(d) $-1, \frac{4}{5}, -\frac{3}{5}, \frac{8}{17}, \ldots$

2.2 Recursion

Explicit definitions are very useful in determining any term in the sequence.

For example: Find a_3, a_{10}, a_{50} of the sequence defined by $\{2^n\}$.

A sequence is defined recursively if each term after the first few terms is defined in terms of some number of previous terms.

Example:

Explicit	Recursive
$\{2n-1\}$	$c_1 = 1$ $c_{n+1} = c_n + 2,\ n \geq 1$

Both define the sequence: 1, 3, 5, 7, …

If you were asked to find c_{100}

$\Downarrow$ *easy* $c_{100} = 2(100) - 1 = 199$	$\Downarrow$ *difficult* Requires the values of: $c_{99}, c_{98}, c_{97}, \cdots$

Note: An objective will be to change recursively defined sequences into explicitly defined sequences. (Difference equations - Later)

Examples:

Write the first four terms of each sequence.

(a) $c_1 = 2 \quad c_{n+1} = c_n + 5,\ n \geq 1$

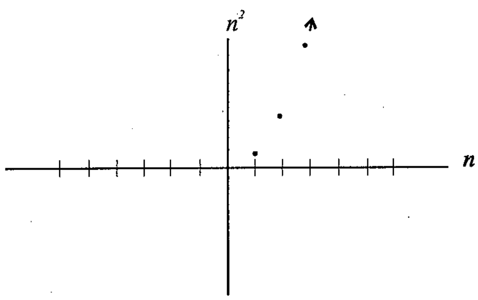
(b) $c_1 = 2 \quad c_{n+1} = c_n^{\ 2} \quad n \geq 1.$

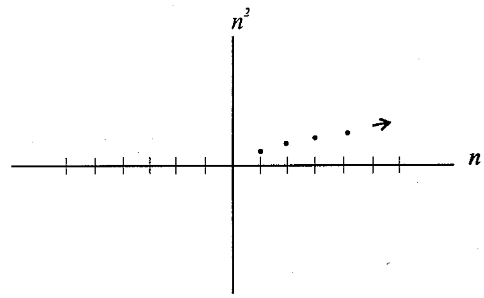
Question: Can you write the above sequences explicitly?

2.3 Limits of Sequences

How do the graphs of the following two sequences differ?

$\{n^2\}$ $\left\{\frac{n}{n+1}\right\}$





Diverges Converges to the number 1

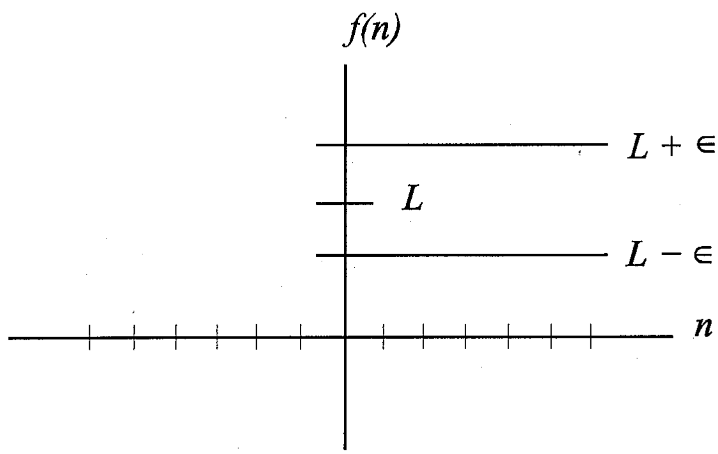
"Definition"

$\lim_{n\to\infty} c_n = L$ provided the values of c_n get closer and closer to L as $n\to\infty$.

[$n\to\infty$ graphically means advancing far to the right]

We say the sequence $\{c_n\}$ converges to L and $\{c_n\}$ is called a convergent sequence.

[Note: Only a finite number of c_n's may be outside the "lane" determined by $(L-\in, L+\in)$ for any value of $\in$.]



If the number L does not exist $\{c_n\}$ diverges and is called a divergent sequence.

Examples:
Determine whether the following sequences have limits. (pg. 12 example 2.1)

(A) $\left\{\frac{1}{2^{n-1}}\right\}$ (B) $\left\{\frac{n^2+2}{n^2+n+2}\right\}$

(C) $\left\{(-1)^n|n-3|\right\}$ (D) $\left\{(-1)^{n+1}\right\}$

More useful facts: [Limit Theorems]
If $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$ then

1. $\lim_{n\to\infty}(ka_n) =$
2. $\lim_{n\to\infty}(a_n + b_n) =$
3. $\lim_{n\to\infty}(a_n \cdot b_n) =$
4. $\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) =$

Examples: (Using the theorems pg. 21 example 2.8)
Find the limits of each sequence, if the limit exists.

(a) $\left\{\left(2-\frac{1}{n^4}\right)\left(\frac{n+3}{3n+1}\right)\right\}$

(b) $\left\{\frac{2^n+1}{3^n-2}\right\}$

(c) $\left\{2^{\frac{1}{n}}+\frac{(-1)^n}{n^6}\right\}$

(d) $\left\{(n-1)\sqrt{\frac{n+1}{2n}}\right\}$

<u>Limits with Radicals</u> (Techniques!)
Example:

1. Find $\lim_{n\to\infty}\left\{\frac{n}{\sqrt{n^2+1}}\right\}$
2. Find $\lim_{n\to\infty}\left\{\frac{\sqrt{3n^2+n+1}}{2n-1}\right\}$
3. Find $\lim_{n\to\infty}\left\{\sqrt{n^2+n}-n\right\}$

Limits of Recursively Defined Sequences
Need theorems!
(To come- The sequence must be bounded, and monotonic for the limit to exist.)
What is a bounded sequence? What is a monotonic sequence?

Monotonic
Definitions:

(a) Increasing if $c_{n+1} > c_n$ for all $n \geq 1$.
(b) Decreasing if $c_{n+1} < c_n$ for all $n \geq 1$.
(c) Non-decreasing if $c_{n+1} \geq c_n$ for all $n \geq 1$.
(d) Non-increasing if $c_{n+1} \leq c_n$ for all $n \geq 1$.

[Note: (c) and (d) are useful if some of the terms are equal. For example: 1,1,2,3,5,...]

If a sequence satisfies any one of the above properties the sequence is monotonic.

Method:
To prove a sequence is monotonic you must show that $c_{n+1} - c_n$ is positive or is negative.
[Calculus students: $f'(x) > 0$ or $f'(x) < 0$.]

Example: Prove that $\left\{2^{\frac{1}{n}}\right\}$ is monotonic.
Solution:

1. (Write the first few terms to decide whether the terms are increasing or decreasing.)

2. (Show that the difference $c_{n+1} - c_n$ agrees with your conjecture in part 1.)

Example; Show that $\left\{\frac{n^2+2}{n^2+n+2}\right\}$ is monotonic.

Bounded

Definitions:

U is an upper bound of $\{c_n\}$ iff. $c_n \le U$ for all $n \ge 1$.

V is a lower bound of $\{c_n\}$ iff. $V \le c_n$ for all $n \ge 1$.

A bounded sequence is a sequence which has a lower and an upper bound.

Method:

(a) To show that U is an upper bound show that $U - c_n \ge 0$ for all $n \ge 1$.

(b) To show that V is a lower bound show that $V - c_n \le 0$ for all $n \ge 1$.

[Note: Usually one of the above is trivial.]

Example:

Show that $\left\{\frac{n}{2^n}\right\}$ is bounded.

Solution

1. (Write the first few terms to decide whether the upper or lower bound is trivial.)

2. (Prove that the remaining bound(s) . You may need to use PMI to do so.)

Example:

Show that $\left\{\frac{n}{n+1}\right\}$ is bounded.

Example 2.9 (pg. 22)

Determine whether $\left\{\dfrac{n+3}{2n+7}\right\}$ is monotonic, bounded, and has a limit.

[Note: Recursively defined sequences often require PMI.]

Example:

Show that the sequence defined by $c_1 = 1 \quad c_{n+1} = 5 + \dfrac{c_n}{10}$, $n \geq 1$ is monotonic and bounded.

Question? What about limits?

<u>Theorem</u>: A bounded monotonic sequence has a limit.

Using this theorem we can find the limit of the sequence in the previous example.

Method:

$$\lim_{n\to\infty} c_{n+1} =$$

Why is boundedness needed?

$\{n^2\}$ is monotonic but not bounded and $\lim_{n\to\infty}\{n^2\} = \infty$

Why must the sequence be monotonic?

$\left\{(-1)^{n+1}\right\}$ is bounded with $U=1$ and $V=-1$ but it is not monotonic.

$\therefore \lim_{n\to\infty}\left\{(-1)^{n+1}\right\}$ does not exist.

Example: #2.10 (pg. 23)

Show that $c_1=1 \;\; c_{n+1}=5+\sqrt{2+c_n}$ for $n\geq 1$ is monotonic and bounded. Hence find its limit.

[Note: If possible, always discuss monotony before bounds.
Why?
If a sequence is monotonic its first term is always a lower (or upper) bound.]

Example: 2.11 (pg.24)

Find the limit of the sequence $c_1=4 \;\; c_{n+1}=\dfrac{c_n}{2}+1,\; n\geq 1$ if it exists.

[Note: Although it is desirable, it is not always possible to determine monotony before boundedness.]

Example: 2.12 (pg. 25)

Find the limit of the sequence $c_1=2 \;\; c_{n+1}=\dfrac{1}{4-c_n},\; n\geq 1$ if it exists.

N.B. Again, you must show that the sequence has a limit before substituting

$$\lim_{n\to\infty}\{c_n\} = L$$

Nonsense:

$c_1=1 \quad c_{n+1}=2c_n+1, n\geq 1$

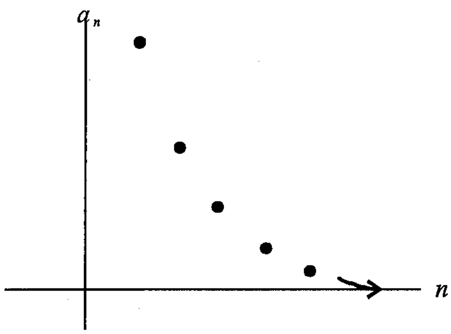
First few terms are: 1, 3, 7, 15, ... (obviously divergent)

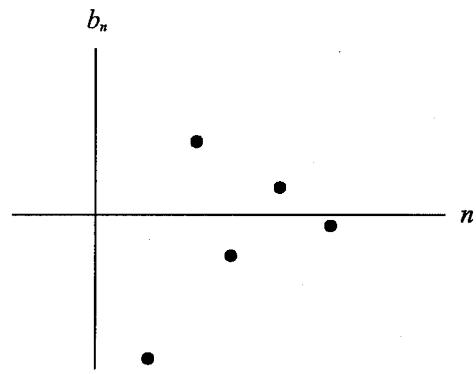
However, if we let $L = 2L + 1$ then $L = -1$, which is silly!

Oscillating Sequences

Sketch each sequence:

(a) $a_n = \dfrac{1}{2^n}$ (b) $b_n = \dfrac{1}{(-2)^n}$





Limit = 0 Limit = 0

In both cases the limit of the sequence is zero, but in a different fashion.

[Note: These sequences are geometric sequences with $r = \dfrac{1}{2}$ and $r = -\dfrac{1}{2}$, respectively.]

$\{a_n\}$ is a monotonic decreasing sequence, whereas $\{b_n\}$ is not monotonic.

$\{b_n\}$ is an oscillating sequence, with terms clustering around 0.

We will only look at <u>recursively defined oscillating sequences</u> and try to find their limits.

[N.B. Since oscillating sequences are not monotonic we cannot apply Theorem 2.2! We need a new theorem to guarantee limits for convergent oscillating recursively defined sequences.]

Theorem 2.3

Suppose a sequence $\{c_n\}$ has the following properties:

1. The differences $c_{n+1} - c_n$ alternate sign. (Hence oscillating!)
2. $\{|c_{n+1} - c_n|\}$ are decreasing
3. $\lim_{n\to\infty} \{|c_{n+1} - c_n|\} = 0$ (Properties 2 and 3 together guarantee existence of a limit.)

Then $\{c_n\}$ converges and $\lim_{n\to\infty} \{c_n\}$ lies between any two successive terms in the sequence.

Example:

Show that the sequence defined by

$$c_1 = 2 \quad c_{n+1} = 2 + \frac{1}{c_n},\ n \geq 1\}$$

converges. Find its limit.

Solution:

Method

1. List the first few terms of the sequence.
2. Find $c_{n+1} - c_n$ and determine that its sign is opposite to the sign of $c_n - c_{n-1}$.
3. Show that the differences $|c_{n+1} - c_n|$ are decreasing with $\lim_{n\to\infty} \{|c_{n+1} - c_n|\} = 0$
4. Find the limit.

Example:

Prove that the recursively defined sequence $c_1 = 4 \quad c_{n+1} = 3 - \frac{c_n}{2},\ n \geq 1$ converges. Find its limit.

Tricky Example #17 (pg.32)

Show that the sequence $c_1 = 1 \quad c_{n+1} = \sqrt{26 - c_n},\ n \ge 1$ is convergent. Find its limit.

Discussion of Solution Attempt:

First few terms are: 1, 5, 4.58, 4.63, … The sequences seems to be oscillating between 4 and 5 for $n \ge 2$.

If we attempt to show oscillation, we get:

$$c_{n+1} - c_n = \left(\sqrt{26-c_n} - \sqrt{26-c_{n-1}}\right)$$

$$= \left(\sqrt{26-c_n} - \sqrt{26-c_{n-1}}\right) \cdot \frac{\left(\sqrt{26-c_n} + \sqrt{26-c_{n-1}}\right)}{\left(\sqrt{26-c_n} + \sqrt{26-c_{n-1}}\right)}$$

$$= \frac{26 - c_n - (26 - c_{n-1})}{\left(\sqrt{26-c_n} + \sqrt{26-c_{n-1}}\right)}$$

$$= \frac{-(c_n - c_{n-1})}{\left(\sqrt{26-c_n} + \sqrt{26-c_{n-1}}\right)}$$

$$= \frac{1}{\left(\sqrt{26-c_n} + \sqrt{26-c_{n-1}}\right)} \cdot \left[-(c_n - c_{n-1})\right]$$

What can be said about the fraction $\dfrac{1}{\left(\sqrt{26-c_n} + \sqrt{26-c_{n-1}}\right)}$?

Especially for $\lim\limits_{n\to\infty} (c_{n+1} - c_n)$

We have a better approach by first proving that other than c_1 the other c_n's are between 4 and 5. This enables us to approximate the above fraction

Conjecture P_n: $4 \le c_n \le 5$, for $n \ge 2$. (Use PMI to prove this conjecture!)

Some examples:

1. (Oct. 2000 #4) Assume that all terms of the sequence

   $c_1 = 10, \quad c_{n+1} = 2\sqrt{16 - c_n}, n \ge 1,$

   satisfy the inequalities $4 \le c_n \le 10$. Do not prove this, assume it.

   (a) Prove that the sequence is oscillating.

   (b) Prove that the sequence is convergent and find its limit.

2. (Oct. 2002 #6) You are given that terms in the sequence

   $c_1 = 1, \quad c_{n+1} = \dfrac{1}{3 - c_n}, n \ge 1$ satisfy $0 \le c_n \le 1$. Do not prove this, assume it.

   Prove that the sequence has a limit and find it. Point out at what steps in your proof you used the given bounds.

3. (Feb. 2001 #6) Prove that the sequence $\left\{ \dfrac{n+2}{(n+1)^2} \right\}_1^\infty$ is monotonic and find its bounds.

4. (Feb. 2001 #7)

   (a) Prove that all terms of the sequence $c_1 = 2, \quad c_{n+1} = 3 + \sqrt{4 - c_n}, n \ge 1$ satisfy $0 \le c_n \le 2$.

   (b) What other information is needed to conclude that the sequence in (a) converges? Do not prove this or find its limit.

5. (Oct. 2000 #3)

   (a) Prove that all terms of the sequence $c_1 = 2, \quad c_{n+1} = \dfrac{1}{3 - c_n}, n \ge 1$ satisfy $0 \le c_n \le 2$.

   (b) What other information is needed in order to conclude that this sequence has a limit? [Do not verify this information or find the limit of the sequence.]

6. An additional example:

   What is the value, if it exists, of x where $x = \sqrt{6 + \sqrt{6 + \sqrt{6 + \sqrt{6 + \cdots}}}}$?