Gradient vector

$$\nabla f(x,y) = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j}$$

$$D_{\vec{v}} f(x,y) = \nabla f(x,y) \cdot \frac{\vec{v}}{\|v\|}$$

<u>Theorem</u> Let f be a function of two (or 3) variables. The maximum value of the directional derivative is when $\vec{v}$ has the same direction as ∇f.

Proof:

$$D_{\vec{v}} f = \nabla f \cdot \vec{v} = \|\nabla f\| \, \|\vec{v}\| \cos\theta$$
$$= \|\nabla f\| \cos\theta$$

$\cos\theta = 1$ if $\vec{v}$ has same direction as ∇f.

Ex $f(x,y) = ye^{x^2}$ what direction should v be in to maximize $D_{\vec{v}}$ at $(1,1)$

$$\nabla f = (2xye^{x^2}, e^{x^2})$$
$$\nabla f(1,1) = (2e, e)$$

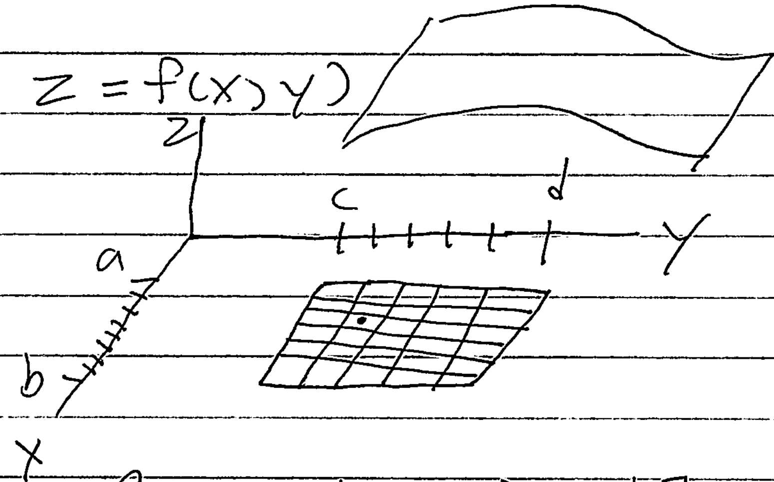
$$v = \left(\frac{2e}{\sqrt{4e^2+e^2}}, \frac{e}{\sqrt{4e^2+e^2}}\right)$$

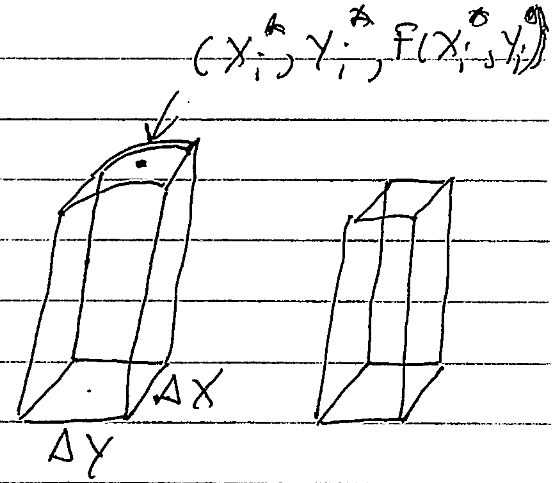
$$= \left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)$$

Double Riemann Sums

for single rieman sums we had $\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$

$= \int_a^b f(x)\,dx.$





$R = [a, b] \times [c, d]$

$\Delta x = \dfrac{(b-a)}{m}$ $\qquad \Delta y = \dfrac{(d-c)}{n}$ $\qquad \Delta A = \Delta x \Delta y$

If $f(x, y) \geq 0$ the volume under $f(x, y) = z$

and over R is approximately

$$\text{Vol} \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i^*, y_j^*)\,\Delta A$$

$$\text{Vol} = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i^*, y_j^*)\,\Delta A$$

$$\iint_R f(x,y)\,dA = V_0]$$

Recall that the average of a function

$f\,[a,b] \to \mathbb{R}$ is

$$f_{av} = \lim_{n\to\infty} \frac{f(x_1^*) + f(x_2^*) + \dots + f(x_n^*)}{n}$$

$$= \frac{1}{b-a} \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

$$= \frac{1}{b-a} \int_a^b f(x)\,dx$$

$\therefore$ The average of $f\,\underbrace{[a,b]\times[c,d]}_{R} \to \mathbb{R}$ is

$$f_{av} = \frac{1}{\text{area}(R)} \iint_R f(x,y)\,dA$$

Properties of double integrals

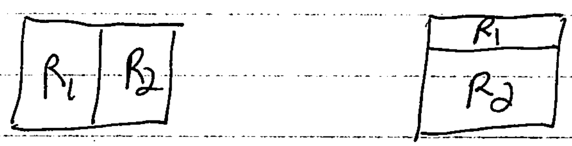
1) $\iint_R f(x,y)+g(x,y)\,dA = \iint_R f(x,y)\,dA + \iint_R g(x,y)\,dA$

2) $\iint_R cf(x,y)\,dA = c\iint_R f(x,y)\,dA$

3) if $g(x,y) \geq f(x,y)$ in R then

$\iint_R g(x,y)\,dA \geq \iint_R f(x,y)\,dA$

4)



$\iint_R f(x,y)\,dA = \iint_{R_1} f(x,y)\,dA + \iint_{R_2} f(x,y)\,dA$

Partial Integration

$\int_c^d f(x,y)\,dy$

hold x fixed, integrate w.r.t y.

Ex $\int_0^1 x^2y^2\,dy = x^2\int_0^1 y^2\,dy = x^2\frac{y^3}{3}\Big|_0^1 = \frac{x^2}{3}$

$$\int_0^1\int_0^1 x^2y^2\,dy\,dx = \int_0^1\left[\int_0^1 x^2y^2\,dy\right]dx$$

$$= \int_0^1 \frac{x^2}{3}\,dx = \frac{x^3}{9}\Big|_0^1 = \frac{1}{9}$$

$$\int_0^1\int_0^1 x^2y^2\,dx\,dy = \int_0^1 \frac{y^2}{3}\,dy = \frac{y^3}{9}\Big|_0^1 = \frac{1}{9}$$

Fubini's Theorem

If $f(x,y)$ is continuous on $R=[a,b]\times[c,d]$

then $\iint_R f(x,y)\,dA = \int_a^b\int_c^d f(x,y)\,dy\,dx$

$$= \int_c^d\int_a^b f(x,y)\,dx\,dy$$