MATH 3GR3 Midterm Test #2 Sample Questions

1. (a) State Lagrange's Theorem.

   Solution: Consult the textbook.

   (b) Let G be a group and suppose that $a \in G$ has order 2. Show that $\{e, a\}$ is a subgroup of G.

   Solution: We need to show that $\{e, a\}$ is closed under the multiplication of the group G, that is closed under taking inverses and that it contains the identity element of G. The last is clear, since e is the identity element of G. Since a has order 2 then $aa = e$ and so $a^{-1} = a$. Since $e^{-1} = e$ then $\{e, a\}$ is closed under taking inverses. Since $aa = ee = e$ and $ea = ae = a$, then this set is also closed under multiplication. Thus $\{e, a\}$ is a subgroup of G.

   (c) Show that if G is a finite group that contains an element a of order 2, then $|G|$ is an even number.

   Solution: From part b) it follows that G has a subgroup H of order 2. Then, by Lagrange's Theorem, 2 divides into $|G|$ and so $|G|$ is an even number.

2. Determine which of the following pairs of groups are isomorphic. Justify your answers to receive credit.

   (a) $\mathbb{Z}$ and $\mathbb{R}$.

   Solution: $\mathbb{Z}$ is a cyclic group, while $\mathbb{R}$ is not, and so these two groups are not isomorphic.

   (b) $\mathbb{Z}$ and $3\mathbb{Z}$.

   Solution: Both of these groups are infinite and cyclic and so they are isomorphic. The map that sends an integer n to the integer $3n$ is an isomorphism from $\mathbb{Z}$ to $3\mathbb{Z}$.

   (c) $\mathbb{Z}_6$ and S_3.

   Solution: $\mathbb{Z}_6$ is an abelian group and S_3 is non-abelian and so these two groups are not isomorphic.

3. Consider the group $GL_3(\mathbb{R})$ of all 3×3 invertible matrices over $\mathbb{R}$, with group operation the usual matrix multiplication, and let

$$H = \left\{ \begin{pmatrix} r & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & r \end{pmatrix} : r \neq 0 \right\} \text{ and } K = \{A \in GL_3(\mathbb{R}) : \det(A) = 1\}.$$

(a) Show that H and K are subgroups of $GL_3(\mathbb{R})$.

Solution: H is closed under addition and inverse, since if $r, s \in \mathbb{R}^*$ then

$$\begin{pmatrix} r & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & r \end{pmatrix} \begin{pmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{pmatrix} = \begin{pmatrix} rs & 0 & 0 \\ 0 & rs & 0 \\ 0 & 0 & rs \end{pmatrix} \in H$$

and

$$\begin{pmatrix} -r & 0 & 0 \\ 0 & -r & 0 \\ 0 & 0 & -r \end{pmatrix} \in H.$$

should be 1/r, not -r down the diagonal.

H contains the identity matrix and so is a subgroup.

Since the product of two matrices that have determinant 1 also has determinant 1 and since the inverse of such a matrix also has determinant 1, then K is closed under product and inverse. K also contains the identity matrix and so is a subgroup.

(b) Show that $GL_3(\mathbb{R})$ is isomorphic to $H \times K$. You may present an explicit isomorphism (with proof) between these two groups to establish the isomorphism, or prove that $GL_3(\mathbb{R})$ is the internal direct product of H and K.

Solution: Since the only matrix that lies in both H and K is the identity matrix, then $H \cap K$ contains only the identity matrix. Since every matrix in H commutes with all matrices, then $AB = BA$ for all $A \in H$ and $B \in K$. To show that $GL_3(\mathbb{R})$ is the internal direct product of H and K we need only show that every matrix in the group is a product of a matrix from H and one from K. If $B \in GL_3(\mathbb{R})$, let $r = \det(B)$. Then the matrix $C = (1/r)B$ has determinant 1 and so is in K. Then

$$B = \begin{pmatrix} 1/r & 0 & 0 \\ 0 & 1/r & 0 \\ 0 & 0 & 1/r \end{pmatrix} C$$

should be r instead of 1/r down the diagonal.

and so B is equal to a product of a matrix from H and a matrix from K. Thus the group is isomorphic to the cartesian product of H and K.

4. Let G be a group and let H and N be subgroups of G. Show that if N is a normal subgroup of G then $HN = \{hn : h \in H \text{ and } n \in N\}$ is a subgroup of G.

   Solution: HN contains the identity element e since it belongs to both H and K. If $a, b \in H$ and $c, d \in N$, then $ac, bd \in HN$ and $(ac)(bd) = a(cb)d = a(bc')d$ for some $c' \in N$, since $Nb = bN$. But $a(bc')d = (ab)(c'd) \in HN$. Thus HN is closed under multiplication. Similarly, $(ab)^{-1} = b^{-1}a^{-1} = a^{-1}b'$ for some $b' \in N$ and so $(ab)^{-1} \in HN$. Thus HN is a subgroup of G.

5. Let G be a group and H a subgroup of G of order n.

   (a) If $g \in G$, show that the set gHg^{-1} is also a subgroup of G and that this subgroup has order n.

   Solution: If $h, k \in H$ then $(ghg^{-1})(gkg^{-1}) = g(hk)g^{-1} \in gHg^{-1}$ and so gHg^{-1} is closed under products. $(ghg^{-1})^{-1} = gh^{-1}g^{-1} \in gHg^{-1}$ and so gHg^{-1} is closed under inverse. This set also contains e since $e = geg^{-1}$ and so this set is a subgroup. The map that sends $h \in H$ to ghg^{-1} is a bijection between H and gHg^{-1} and so these two sets have the same size.

   (b) Suppose that H is the only subgroup of G that has order n. Show that H is a normal subgroup of G.

   Solution: From par a) it follows that for all $g \in G$, $gHg^{-1} = H$, since both sets are subgroups of order n. But this condition is equivalent to H being a normal subgroup.