MATH 3GR3 Assignment #1

Due: Monday, September 27, by 11:59pm

Upload your solutions to the Avenue to Learn course website.

Detailed instructions will be provided on the course website.

NOTE: Question #5 has been slightly modified (to make the solution shorter and slightly easier).

Please read the following statement on collaboration on homework:

Limited collaboration in planning and thinking through solutions to homework problems is allowed, but no collaboration is allowed in writing up solutions. It is permissible to discuss general aspects of the problem sets with other students in the class, but each person should hand in his/her own copy of the solutions. By general aspects I mean you can say things like, "Did you use Lagrange's Theorem for question 1?" Anything more detailed than this is not acceptable.

Violation of these rules may be grounds for giving no credit for a homework paper and also for serious disciplinary action.

In presenting your solutions, I will be looking for well written, comprehensible answers. Please don't shy away from using complete English sentences to explain your work, and please be careful how you use quantifiers. Every statement you write down should assert something, and should be used somehow to help solve the problem at hand.

1. Let X and Y be nonempty sets and let $f : X \to Y$ be a function. Show that the relation $a \sim b$ if and only if $f(a) = f(b)$ is an equivalence relation on the set X. Show that there is a one-to-one correspondence between the equivalence classes of the relation $\sim$ and the range of the function f.

2. Determine which of the following relations are equivalence relations. For those that are, describe the partition that arises from them.

   (a) The relation $R = \{(m, n) : mn \geq 0\}$ on the set $\mathbb{Z}$.

   (b) The relation $R = \{(m, n) : mn > 0\}$ on the set $\mathbb{Z} \setminus \{0\}$.

(c) For $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$, $(x_1, y_1) \sim (x_2, y_2)$ if and only if $x_1^2 + y_1^2 = x_2^2 + y_2^2$

3. Let G be a group and H a subgroup of G. Define the following relation on G:

   $a \sim b$ if and only if $ab^{-1} \in H$.

   Show that $\sim$ is an equivalence relation on G.

4. Write out the Cayley table for the group of symmetries of a regular pentagon. **BONUS:** Find all of the subgroups of this group.

5. Suppose that $G = \{e, a, b, c\}$ is a four element group. Write out the possible Cayley tables for G, assuming that the element e is the identity element of the group **and that** $ab = c$. Note that $ab = e$ is also a possibility, but you don't need to deal with this case in your solution.

6. Let G be a group and $a \in G$. Suppose that m and n are relatively prime integers such that $a^m = e$. Show that there is some element $b \in G$ with $a = b^n$

7. Determine if the set $G = \{(x, y) \in \mathbb{R}^2 : y \neq 0\}$ along with the operation $(a, b) * (c, d) = (ad + bc, bd)$ is a group.

8. Let n be a positive integer. Prove that there are only finitely many distinct groups G on the n-element set $\{0, 1, 2, \ldots, n-1\}$, i.e., there are only finitely many Cayley tables for the set $\{0, 1, 2, \ldots, n-1\}$. Provide some upper bound, as a function of n, on the number of such groups. Note that in Question #4 you've essentially shown that once the identity element for a group on a 4-element set has been selected, there aren't that many Cayley tables. This is a very special case, and things are usually more complicated for larger values of n.

9. Let G be a group that satisfies the following property:

   If x, y, $z \in G$ with $xy = zx$, then $y = z$.

   Show that G is an abelian group.

10. Go to `http://abstract.ups.edu/aata/aata.html` and do the SAGE tutorials for Chapters 1 and 2. Then perform the following calculations using SAGE. It is probably easiest to enter and execute your SAGE commands using SageCell (`https://sagecell.sagemath.org/`). A SageCell window has been set up at the bottom of the course webpage and can be used for this assignment (and to perform other SageMath calculations).

    To submit your calculations, either take a screenshot (or maybe a picture) of the webpage that contains them or include a copy of the link (i.e., URL) that is produced by the SageCell "share" button, or just copy and paste your commands and the results into the document that you upload to Avenue to Learn. For a more powerful environment to run your SAGE commands, you can use COCALC (`https://cocalc.com/`).

    You will need to know your McMaster student number. Let's denote it by N.

    (a) Compute the remainder when N (your student number) is divided by $V = 77115025$ (Dr. Valeriote's **very old** student number).

    (b) Compute the greatest common divisor of N and V and find integers a and b such that the gcd is equal to $aN + bV$.

    (c) Find the prime factorization of N and

    (d) the smallest prime number p with $p > N$.

Supplementary problems from the textbook (not to be handed in)

- From Chapter 1, questions 21, 25, 26
- From Chapter 2, questions 16, 18, 24, 26
- From Chapter 3, questions 2, 3, 6, 7, 15, 17