MATH 3GR3 Assignment #1
Due: Friday, September 22, 11:59pm
Upload your solutions to the Avenue to Learn course website.
Detailed instructions will be provided on the course website.

Important notes:

- In this course, all work submitted for grading must be your own. Limited collaboration in planning and thinking through solutions to homework problems is allowed, but no collaboration is allowed in writing up solutions. It is permissible to discuss general aspects of the problem sets with other students in the class, but each person should hand in their own copy of the solutions. By general aspects I mean you can say things like, "Did you use Lagrange's Theorem for question 1?" Anything more detailed than this is not acceptable.

  Violation of these rules may be grounds for giving no credit for a homework paper and also for serious disciplinary action.

- You may not submit solutions found on the internet or generated using AI tools, such as ChatGPT. Please carefully read over the course announcement and the Senate Policy on Academic Integrity. It is your responsibility to know and understand this policy. If you have any questions about this, please contact Dr. Valeriote.

- In presenting your solutions, I will be looking for well written, comprehensible answers. Please don't shy away from using complete English sentences to explain your work, and please be careful how you use quantifiers. Every statement you write down should assert something, and should be used somehow to help solve the problem at hand.

- To submit your solutions, you will need to upload to the course Avenue To Learn site, a **single pdf file** that contains all of your solutions to this assignment. No other file formats will be accepted. Free online tools are readily available to convert many common file formats to the pdf format and to merge pdf files. If you need assistance with this, please contact the course TA or Dr. Valeriote before the assignment deadline.

1. Determine which of the following relations are equivalence relations. For those that are, describe the partition that arises from it.

   (a) $R = \{(a, b) \in \mathbb{Z}^2 : ab > 0\}$
   (b) $R = \{(a, b) \in \mathbb{R}^2 : |a| = |b|\}$
   (c) For $a, b \in \mathbb{N}$, $a \sim b$ if and only if the number of digits in the decimal representations of a and b are the same
   (d) For $a, b \in \mathbb{R}$, $a \sim b$ if and only if $|a - b| \leq 2$.

2. Let $G = \{e, x, y\}$ be any group with three elements. Produce the Cayley (multiplication) table for G.

3. Let $G = \{e, x, y, z\}$ be a group with four elements, with e the identity element. Show that there are exactly two possibilities for the Cayley table for G, up to a rearrangement of the elements. This means that any Cayley table that can be obtained from another one by, for example, interchanging y with z in all places in the table, should be considered as the same table.

4. Prove that every non-abelian group G has at least six elements, i.e., every group of size 5 or less is abelian.

5. Describe the set of symmetries of a circular disc. In particular, show that this set is infinite.

6. Describe the set of symmetries of a regular tetrahedron. You do not need to provide the multiplication table for the corresponding group.

7. Let G be a set and $\circ$ a binary operation on G that satisfies the following properties:

   (a) $\circ$ is associative,
   (b) There is an element $e \in G$ such that $e \circ a = a$ for all $a \in G$,
   (c) For every $a \in G$, there is some $b \in G$ such that $b \circ a = e$.

   Prove that $(G, \circ)$ is a group.

8. Let G be a group such that $a^2 = e$ for all $a \in G$. Prove that G is abelian.

9. Let G be a finite group. Show that the number of elements a of G with the property that $a^3 = e$ is odd. Show that the number of elements a of G with the property that $a^2 \neq e$ is even.

10. Go to `http://abstract.ups.edu/aata/aata.html` and do the SAGE tutorials for Chapters 1 and 2. Then perform the following calculations using SAGE. It is probably easiest to enter and execute your SAGE commands using SageCell (`https://sagecell.sagemath.org/`). A SageCell window has been set up at the bottom of the course webpage and can be used for this assignment (and to perform other SageMath calculations).

    To submit your calculations, either take a screenshot (or maybe a picture) of the webpage that contains them or include a copy of the link (i.e., URL) that is produced by the SageCell "share" button, or just copy and paste your commands and the results into the document that you upload to Avenue to Learn. For a more powerful environment to run your SAGE commands, you can use COCALC (`https://cocalc.com/`).

    You will need to know your McMaster student number. Let's denote it by N.

    (a) Compute the remainder when N (your student number) is divided by $V = 77115025$ (Dr. Valeriote's **very old** student number).

    (b) Compute the greatest common divisor of N and V and find integers a and b such that the gcd is equal to $aN + bV$.

    (c) Find the prime factorization of N and

    (d) the smallest prime number p with $p > N$.

Supplementary problems from the textbook (not to be handed in)

- From Chapter 1, questions 21, 25, 26
- From Chapter 2, questions 16, 18, 24, 26
- From Chapter 3, questions 2, 3, 6, 7, 15, 17