MATH 3GR3 Assignment #2
Due: Monday, 18 October by 11:59pm

1. Show that if G is an abelian group and a, $b \in G$ both have finite order, then so does the element ab. Find an example of a group G and two elements a, $b \in G$ both of finite order such that ab has infinite order.

2. Find all of the subgroups of $\mathbb{Z}_{12}$ and find all of the generators of this group.

3. Produce the Cayley table of the group $U(12)$. Is this group cyclic?

4. Let G be a finite cyclic group. Show that if H is a subgroup of G then $|H|$ divides $|G|$. Conversely, show that if k is a natural number such that k divides $|G|$ then there is a subgroup of G of order k.

5. Let H and K be subgroups of the group G. Show that $H \cap K$ is a subgroup of G. Show that the subset $HK = \{h \cdot k : h \in H, k \in K\}$ is not necessarily a subgroup, by finding an example that illustrates this.

6. Consider the following two elements of S_7:

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 6 & 7 & 4 & 3 & 1 & 5 & 2 \end{pmatrix}, \tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 3 & 4 & 5 & 7 & 6 & 2 \end{pmatrix}.$$

   (a) Decompose σ and τ into cycles.

   (b) Compute $\sigma\tau$ and $\tau\sigma$.

   (c) Compute the order of σ, τ, $\sigma\tau$, and $\tau\sigma$.

   (d) Determine the signs of σ, τ, $\sigma\tau$, and $\tau\sigma$.

7. What are the possible orders of elements in the group S_8? For each one, find an element of that order.

8. Show that every element σ of the group S_n can be written as a product of transpositions of the form $(i, i+1)$, where $1 \leq i < n$.

9. Let G be the group of symmetries of the circular disk of radius 1. Show that G contains elements of every finite order and that it contains elements of infinite order.

Supplementary problems from the textbook
(not to be handed in)

- From Chapter 3, questions 34, 35, 49, 50
- From Chapter 4, questions 1, 2, 3, 10, 23, 24, 25, 30, 39