MATH 3GR3 Assignment #3
Due: Friday, October 27, 11:59pm
Upload your solutions to the Avenue to Learn course website.
Detailed instructions will be provided on the course website.

1. Consider the following two elements of $S_7$:

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 6 & 7 & 4 & 3 & 1 & 5 & 2 \end{pmatrix}, \tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 3 & 4 & 5 & 7 & 6 & 2 \end{pmatrix}.$$

   (a) Decompose $\sigma$ and $\tau$ into cycles.
   (b) Compute $\sigma\tau$ and $\tau\sigma$.
   (c) Compute the order of $\sigma$, $\tau$, $\sigma\tau$, and $\tau\sigma$.
   (d) Determine the signs of $\sigma$, $\tau$, $\sigma\tau$, and $\tau\sigma$, i.e., determine if they are even or odd permutations.

2. Let $\sigma = (a_1, a_2, \ldots, a_m)$ and $\tau = (b_1, b_2, \ldots, b_n)$ be cycles of length $m$ and $n$ respectively in the group $S_X$ for some set $X$. Suppose that

   - $\{a_1, a_2, \ldots, a_m\} \neq \{b_1, b_2, \ldots, b_n\}$ and that
   - $\{a_1, a_2, \ldots, a_m\} \cap \{b_1, b_2, \ldots, b_n\} \neq \emptyset$.

   So $\sigma$ and $\tau$ are not disjoint, but the sets of elements that they permute are different.

   Show that $\sigma\tau \neq \tau\sigma$.

   HINT: Since the elements being permuted are elements of some set $X$, you may assume that $X$ is just $\{1, 2, \ldots, k\}$ for some large enough $k$ (bigger than $n + m$), that $\sigma = (1, 2, \ldots, m)$, that the element 1 is equal to one of the $b_i$'s and that $\tau$ is of the form $(1, b_2, \ldots, b_n)$ with $b_2 > m$. To use this hint, you should provide a justification for why this reduction of the general case to this specific case is valid.

3. (a) We have seen that $D_3$, the group of symmetries of the equilateral triangle, is not abelian. Show that for $n > 2$, the dihedral group $D_n$ is not abelian.

(b) Find all elements $a$ of the group $D_8$ that commute with every element of $D_8$, i.e., find $\{a \in D_8 \colon ax = xa \text{ for all } x \in D_8\}$. Is this set a subgroup of $D_8$?

4. For each pair of group $G$ and subgroup $H$, describe the left and right cosets of $H$ in $G$:

   (a) $G = D_n$, $H = \langle s \rangle$.

   (b) $G = GL_2(\mathbb{Q})$, $H = SL_2(\mathbb{Q})$.

   (c) $G = A_4$, $H = \{(1), (123), (132)\}$.

   (d) $G = \mathbb{Z}_{12}$, $H = \langle 10 \rangle$.

5. Recall that $GL_2(\mathbb{R})$ is the group of all $2 \times 2$ invertible matrices with real entries. Let

   - $G = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \in GL_2(\mathbb{R}) \ : \ ac \neq 0 \right\}$
   - $H = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \ : \ x \in \mathbb{R} \right\}$

   (a) Show that $G$ is a subgroup of $GL_2(\mathbb{R})$ and that $H$ is a subgroup of $G$.

   (b) Show that every left coset of $H$ in the group $G$ is equal to a right coset of $H$ in $G$.

   (c) Show that for $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}, \begin{pmatrix} u & v \\ 0 & w \end{pmatrix} \in G$, the two left cosets $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} H$ and $\begin{pmatrix} u & v \\ 0 & w \end{pmatrix} H$ are equal if and only if $a = u$ and $c = w$.

   (d) $H$ is also a subgroup of $GL_2(\mathbb{R})$ since it has been established in (a) that $H$ is a subgroup of $G$ and $G$ is a subgroup of $GL_2(\mathbb{R})$. Are the left cosets of $H$ in $GL_2(\mathbb{R})$ equal to right cosets of $H$ in $GL_2(\mathbb{R})$?

6. Let $G$ be a group of order $343 = 7^3$. Show that $G$ contains an element of order 7.

7. Let $G$ be a finite group that contains elements of order 1 through 10. What is the smallest possible order of $G$? Provide an example of a group with this property having this order.

8. Read over the SageMath tutorials at the ends of Chapters 4, 5, and 6 and perform the following calculations. To submit your calculations, either take a screenshot (or maybe a picture) of the webpage that contains them or include a copy of the link (i.e., URL) that is produced by the SageCell "share" button, or just copy and paste your commands and the results into the document that you upload to Avenue to Learn.

   (a) Produce the alternating group $A_7$ using the "AlternatingGroup" command and then create and list the elements of the cyclic subgroup of $A_7$ that is generated by the (even) permutation

   $$(1,2,3)(4,7)(5,6).$$

   You will need to use the "subgroup" function to create the subgroup.

   (b) Produce all of the subgroups of the group $A_4$ and list all of the orders of these subgroups. Do not list the subgroups, just their orders. Verify that $A_4$ has no subgroup of order 6 (you don't need to include this in your solution, just look at your list to verify this. We claimed in the lectures that even though $|A_4| = 12$, $A_4$ doesn't have a subgroup of order 6, a divisor of 12. A description of how to do this can be found in the Subgroups subsection of Section 6.6 of the textbook.

## Supplementary problems from the textbook (not to be handed in)

- From Chapter 5, questions 1, 3, 6, 9, 23, 30, 31
- From Chapter 6, questions 3, 5, 10, 16, 17