MATH 3GR3 Assignment #4
Due: Friday, 10 November, by 11:59pm

1. Let G be a group with $|G| < 300$. Suppose that G has a subgroup H with order 24 and a subgroup K with order 54. Determine the exact value of $|G|$.

2. For each pair of groups G and H, determine if they are isomorphic:
   (a) $G = \mathbb{R}^*$, $H = \mathbb{C}^*$.
   (b) $G = U(14)$, $H = U(18)$.
   (c) $G = \mathbb{Z}$, $H = \mathbb{R}$.
   (d) $G = \mathbb{Z}_{16}$ and $H = \mathbb{Z}_4 \times \mathbb{Z}_4$.

3. Show that the group $U(4) \times U(5)$ is isomorphic to the group $U(20)$.

4. Let G_1 and G_2 be groups and suppose that H_1 is a subgroup of G_1 and H_2 is a subgroup of G_2. Show that $H_1 \times H_2$ is a subgroup of $G_1 \times G_2$. Find a subgroup of $\mathbb{Z}_4 \times \mathbb{Z}_2$ that is not of this form.

5. Let G be a group, H a subgroup of G, and $g \in G$.
   (a) Show that the map $f : G \to G$ defined by $f(x) = gxg^{-1}$ is an isomorphism from G to G.
   (b) Show that the set $gHg^{-1} = \{ghg^{-1} : h \in H\}$ is a subgroup of G.
   (c) Show that $\displaystyle\bigcap_{g\in G} gHg^{-1}$ is a normal subgroup of G.

6. Show that every group of order 4 is isomorphic to $\mathbb{Z}_4$ or to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

7. (a) Show that $H = \{id, (1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\}$ is a normal subgroup of A_4.
   (b) Show that H is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.
   (c) Produce the Cayley table of the quotient group A_4/H.
   (d) Show that H is the only non-trivial normal subgroup of A_4.

8. Let G be a group and define $Z(G)$ to be the subset $\{g \in G : gx = xg \text{ for all } x \in G\}$.
   (a) Show that $Z(G)$ is a normal subgroup of G.

(b) Compute $Z(D_4)$.

(c) Compute the Cayley table for the quotient group $D_4/Z(D_4)$.

9. Let G be a group and let $D = \{(g, g) : g \in G\}$, a subgroup of $G \times G$.

   (a) Show that D is isomorphic to the group G.

   (b) Show that D is a normal subgroup of the group $G \times G$ if and only if G is an abelian group.

10. Read over the SageMath tutorials at the ends of Chapters 6 and 9 and perform the following calculations. To submit your calculations, either take a screenshot (or maybe a picture) of the webpage that contains them or include a copy of the link (i.e., URL) that is produced by the SageCell "share" button, or just copy and paste your commands and the results into the document that you upload to Avenue to Learn.

    (a) Compute $\phi(2023)$, where ϕ is the Euler ϕ-function.

    (b) Produce the cyclic group G of order 40 using the "CyclicPermutationGroup" function and the group D_{20} and then use the "is_isomorphic" function to determine if these two groups are isomorphic.

Supplementary problems from the textbook (not to be handed in)

- From Chapter 9, questions 16, 19, 22, 23, 48, 52
- From Chapter 10, questions 1, 2, 4, 5, 6, 9, 13