MATH 3GR3 Assignment #5 Solutions
Due: Friday, November 24 by 11:59pm

1. Suppose that G is a cyclic group and that N is a subgroup of G. Show that G/N is also a cyclic group.

   Solution: Let g be a cyclic generator of G. Then the element gN is a cyclic generator of G/N. To see this, let $C \in G/N$. Then C is a left coset of N and so $C = hN$ for some $h \in G$. Since $G = \langle g \rangle$ then $h = g^n$ for some $n \in \mathbb{Z}$. Then $(gN)^n = g^n N = hN = C$ and so $C \in \langle gN \rangle$. This shows that every element of G/N is equal to some power of gN and so G/N is cyclically generated by gN.

2. Let G and H be groups and let $M \trianglelefteq G$ and $N \trianglelefteq H$.

   (a) Show that the map $f : G \times H \to G/M \times H/N$ defined by $f((g,h)) = (gM, hN)$ is an onto group homomorphism.

   Solution: To see that this map is a group homomorphism, let g_1, $g_2 \in G$ and h_1, $h_2 \in H$. Then

$$\begin{aligned} f((g_1,h_1)(g_2,h_2)) &= f((g_1g_2, h_1h_2)) = (g_1g_2M, h_1h_2N) \\ &= (g_1Mg_2M, h_1Nh_2N) = (g_1M, h_1N)(g_2M, h_2N) \\ &= f((g_1,h_1))f((g_2,h_2)). \end{aligned}$$

   This map is onto since any element of $G/M \times H/N$ is of the form (gM, hN) for some $g \in G$ and $h \in H$. The element (g,h) in $G \times H$ is mapped to (gM, hN) under f.

   (b) Show that the kernel of f is $M \times N$.

   Solution: The element (g,h) will be in the kernel of f if and only if $f((g,h)) = (M,N)$, the identity element of $G/M \times H/N$. Since $f((g,h)) = (gM, hN)$ then (g,h) will be in the kernel if and only if $(gM, hN) = (M,N)$, which is equivalent to $gM = M$ and $hN = N$, which is equivalent to $g \in M$ and $h \in N$. Thus the kernel is equal to $M \times N$.

(c) Prove that $(G \times H)/(M \times N)$ is isomorphic to $G/M \times H/N$. (Hint: Use the First Isomorphism Theorem.)

Solution: In parts (a) and (b) we've established that f is an onto homomorphism with kernel equal to $M \times N$. By the First Isomorphism Theorem, $(G \times H)/\ker(f)$ is isomorphic to the image of f and so $(G \times H)/(M \times N)$ is isomorphic to $G/M \times H/N$.

3. Determine which of the following maps are group homomorphisms. For those that are, compute their kernels.

(a) For $n > 1$, $f : \mathbb{Z} \to \mathbb{Z}_n$ is defined by $f(m) = [m]_n$.

Solution: This is a group homomorphism, since for m, $k \in \mathbb{Z}$,

$$f(m+k) = [m+k]_n = [m]_n + [k]_n = f(m) + f(k).$$

The kernel of this map is the set of all m with $[m]_n = [0]_n = n\mathbb{Z}$. (It follows that since this map is onto, then by the First Isomorphism Theorem, $\mathbb{Z}/n\mathbb{Z}$ is isomorphic to $\mathbb{Z}_n$. In fact the groups are equal.)

(b) $f : \mathbb{R}^* \to \mathbb{Z}_2$ defined by $f(r) = 0$ if $r > 0$ and $f(r) = 1$ if $r < 0$.

Solution: To prove that $f(r \cdot s) = f(r) + f(s)$, we can consider 4 cases:

- $r, s > 0$: then $r \cdot s > 0$ and so $f(r \cdot s) = 0 = 0 + 0 = f(r) + f(s)$.
- $r, s < 0$: then $r \cdot s > 0$ and so $f(r \cdot s) = 0 = 1 + 1 = f(r) + f(s)$.
- $r > 0$ and $s < 0$: then $r \cdot s < 0$ and so $f(r \cdot s) = 1 = 0 + 1 = f(r) + f(s)$.
- $r < 0$ and $s > 0$: then $r \cdot s < 0$ and so $f(r \cdot s) = 1 = 1 + 0 = f(r) + f(s)$.

Thus f is a group homomorphism. The kernel is the set $\{r : r > 0\}$.

(c) $f : \mathbb{Q} \to \mathbb{Q}$ defined by $f(q) = |q|$.

Solution: This is not a group homomorphism since the equation $f(q_1 + q_2) = f(q_1) + f(q_2)$ does not hold for all q_1, $q_2 \in \mathbb{Q}$. A counter example is

$$f(1 + (-1)) = f(0) = |0| = 0 \neq 2 = 1 + 1 = f(1) + f(-1).$$

4. Let F be the group of all functions $f : \mathbb{Z} \to \mathbb{Z}$ with group operation $+$ defined by $(f + g)(x) = f(x) + g(x)$. For this problem you do not need to show that F is a group under this operation. Use the First Isomorphism Theorem to show that $N = \{f \in F \colon f(3) = 0\}$ is a normal subgroup of F and that F/N is isomorphic to $\mathbb{Z}$.

   Solution: To show that N is a normal subgroup of F and to establish the isomorphism, it suffices to find an onto homomorphism ϕ from F to $\mathbb{Z}$ that has kernel equal to N. Both results will then follow from the First Isomorphism Theorem. The function $\phi : F \to \mathbb{Z}$ defined by $\phi(f) = f(3)$ is a map from F to $\mathbb{Z}$. It is onto, since for any integer n, $\phi(f) = n$, where f is the constant function on $\mathbb{Z}$ that takes on the value n. ϕ is a homomorphism since

   $$\phi(f + g) = (f + g)(3) = f(3) + g(3) = \phi(f) + \phi(g).$$

   The kernel of ϕ is $\{f \in F \mid \phi(f) = 0\} = N$, as required.

5. Let $N = \{-1, 1\}$, a subgroup of the group $\mathbb{Q}^*$, and let $\mathbb{Q}^+$ be the subgroup of $\mathbb{Q}^*$ consisting of all positive rational numbers. Use the First Isomorphism Theorem to show that $\mathbb{Q}^*/N$ is isomorphic to $\mathbb{Q}^+$ by constructing a surjective homomorphism from $\mathbb{Q}^*$ to $\mathbb{Q}^+$ that has kernel N.

   Solution:

   As in the previous question, it suffices to find an onto homomorphism ϕ from $\mathbb{Q}^*$ to $\mathbb{Q}^+$ that has kernel equal to N. The following map will work: $\phi(q) = |q|$, so ϕ is the " absolute value function". The kernel of ϕ is the set of q with $\phi(q) = 1$, the identity element of $\mathbb{Q}^+$. But this is just the set N, as required. Using the properties of the absolute value function, it is elementary to show that ϕ is an onto homomorphism, so by the First Isomorphism Theorem, $\mathbb{Q}^*/N$ is isomorphic to $\mathbb{Q}^+$.

6. Let G be a group and N a normal subgroup of G. Show that if $aba^{-1}b^{-1} \in N$ for all a, $b \in G$, then the factor group G/N is abelian. Is the converse true?

   Solution: The following argument establishes the result and also shows that the converse is true. Given a group G and normal subgroup N with the stated property,

- G/N is abelian if and only if
- for all a, $b \in G$, $(aN)(bN) = (bN)(aN)$, if and only if
- for all a, $b \in G$, $(abN) = (baN)$ if and only if
- for all a, $b \in G$, $(ab)(ba)^{-1} \in N$ (by Lemma 6.3), if and only if
- for all a, $b \in G$, $aba^{-1}b^{-1} \in N$.

Supplementary problems from the textbook (not to be handed in)

From Chapter 11, questions 2, 3, 4, 6, 8, 9, 10, 13, 16