MATH 3GR3 Assignment #6
Due: Wednesday, December 6 by 11:59pm

1. Consider the ring $\mathbb{Z}_{20}$. List all of the ideals of this ring. List all of the units of this ring.

2. For each pair of rings, determine if they are isomorphic.

   (a) $\mathbb{R}$ and $\mathbb{C}$.

   (b) $\mathbb{Z}$ and $\mathbb{Z}[i]$.

3. Show that the map $f : \mathbb{C} \to M_{2\times 2}(\mathbb{R})$ defined by

$$f(a+bi) = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$$

   is a one-to-one homomorphism from the ring of complex numbers to the ring of 2×2 matrices with real entries.

4. Let R be a commutative ring with identity and suppose that I and J are ideals of R. Show that $I \cap J$ is also an ideal of R. If I and J are prime ideals of R will $I \cap J$ always be a prime ideal of R?

5. Let

$$I = \{a_0 + a_1x + a_2x^2 + \cdots + a_nx^n \in \mathbb{Z}[x] \mid a_0 \text{ is even}\}.]$$

   (a) Show that I is an ideal of $\mathbb{Z}[x]$.

   (b) Show that $\mathbb{Z}[x]/I$ is isomorphic to $\mathbb{Z}_2$.

   (c) Prove that I is a maximal ideal of $\mathbb{Z}[x]$.

6. Let $R = \mathbb{Z}[x]$ and let I be the set of polynomials of $\mathbb{Z}[x]$ whose terms have degree at least 2, plus the constant 0 polynomial. So, members of I are of the form $a_2x^2 + a_3x^3 + \cdots + a_nx^n$ for some $n \geq 2$ and integers a_i.

   (a) Show that I is an ideal of R. Hint: Show that $I = \langle x^2 \rangle$.

   (b) Show that the polynomials $3 + 5x + x^3 + x^5$ and $3 + 5x - x^4$ are in the same coset of I and give a general condition for when two polynomials $p(x)$ and $q(x)$ lie in the same coset of I.

(c) Show that R/I consists of the elements $(a + bx) + I$ for $a, b \in \mathbb{Z}$.

(d) Describe the addition and multiplication operations on R/I.

(e) Is R/I an integral domain? (this is the same as asking if I is a prime ideal.)

Bonus:

(a) Compute the remainder when the polynomial $8x^5 - 18x^4 + 20x^3 - 25x^2 + 20$ is divided by $4x^2 - x - 2$. Both polynomials are members of the polynomial ring $\mathbb{Q}[x]$.

(b) Compute the remainder when the polynomial $3x^4 + x^3 + 2x^2 + 1$ is divided by $x^2 + 4x + 2$. Both polynomials are members of the polynomial ring $\mathbb{Z}_5[x]$.

Supplementary problems from the textbook (not to be handed in)

- From Chapter 16, questions 1, 2, 4, 12, 16, 18, 23, 26, 30, 33, 36
- From Chapter 17, questions 3, 6, 13, 16, 26, 29