Proposition 11.4: Let $\phi: G_1 \to G_2$ be a group homomorphism. Then

① If e is the identity element of G_1, then $\phi(e)$ is the identity element of G_2.

② If $g \in G_1$, then $\phi(g)^{-1} = \phi(g^{-1})$.

③ If H_1 is a subgroup of G_1, then $\phi(H_1)$ is a subgroup of G_2.

④ If H_2 is a subgroup of G_2, then $\phi^{-1}(H_2) = \{g \in G_1 \mid \phi(g) \in H_2\}$ is a subgroup of G_1.

If H_2 is a normal subgroup of G_2, then $\phi^{-1}(H_2)$ is a normal subgroup of G_1.

Proof of ④.

$e \in \phi^{-1}(H_2)$ by ①.

Let $a, b \in \phi^{-1}(H_2)$.

Then $\phi(a) = h_1 \in H_2$ and $\phi(b) = h_2 \in H_2$ for some h_1, h_2.

Then $\phi(ab) = \phi(a)\phi(b)$
$= h_1 h_2 \in H_2$

So $\phi(ab) \in H_2$, and so $ab \in \phi^{-1}(H_2)$.

By ②, $\phi^{-1}(H_2)$ is closed under taking inverses.

So, $\phi^{-1}(H_2)$ is a subgroup of G_1

If $H_2 \trianglelefteq G_2$, then,

let $g \in G_1$ and

show $g\phi^{-1}(H_2)g^{-1} \subseteq \phi^{-1}(H_2)$.

Let $g_1 \in \phi^{-1}(H_2)$,

Then $\phi(g_1) \in H_2$.

Show $\phi(gg_1g^{-1}) \in H_2$

$\phi(gg_1g^{-1}) = \phi(g)\underbrace{\phi(g_1)}_{\in H_2}\phi(g)^{-1} \in H_2$,

Since H_2 is normal.

So, $gg_1g^{-1} \in \phi^{-1}(H_2)$ $\boxed{}$

Exercise: Let $\phi: G_1 \to G_2$ be an onto homomorphism, and H_1 a normal subgroup of G_1.

Show $\phi(H_1)$ is a normal subgroup of G_2.

Show that, if ϕ is not onto then $\phi(H_1)$ need not be a normal subgroup of G_2.

Observe: If e' is the identity element of G_2, then $\{e'\} \trianglelefteq G_2$.

By ④, $\phi^{-1}(\{e'\}) \trianglelefteq G_1$

We call this normal subgroup of G_1 the kernel of ϕ

Definition: For $\phi: G_1 \to G_2$ a homomorphism, the kernel of ϕ, denoted $\ker(\phi)$, is

the normal subgroup $\phi^{-1}(\{e'\})$

$$\ker(\phi) = \{g \in G_1 \mid \phi(g) = e'\},$$

where e' is the identity element of G_2

eg: ① $\phi : S_n \to \mathbb{Z}_2$, $\phi(\sigma) = \begin{cases} 0, & \text{if } \sigma \text{ is even} \\ 1 & \text{if } \sigma \text{ is odd} \end{cases}$

$$\ker(\phi) = \{\sigma \in S_n \mid \phi(\sigma) = 0\}$$
$$= \{\sigma \in S_n \mid \sigma \text{ is even}\}$$
$$= A_n$$

So $A_n \trianglelefteq S_n$ $([S_n : A_n] = 2)$

- $\det: GL_n(\mathbb{R}) \to \mathbb{R}^*$

$$\ker(\det) = \{A \in GL_n(\mathbb{R}) \mid \det(A) = 1\}$$
$$= SL_n(\mathbb{R}).$$

So $SL_n(\mathbb{R}) \trianglelefteq GL_n(\mathbb{R})$.

- $\phi: \mathbb{Z} \to \mathbb{Z}_n$

$$\phi(k) = [k]_n \in \mathbb{Z}_n$$

$$\ker(\phi) = \{k \in \mathbb{Z} \mid [k]_n = [0]_n\}$$
$$= [0]_n$$
$$= n\mathbb{Z}$$

>1.

<u>Claim</u>: If $N \trianglelefteq G$, then $N = \ker(\phi)$ for some homomorphism $\phi: G \to H$ for some group H.

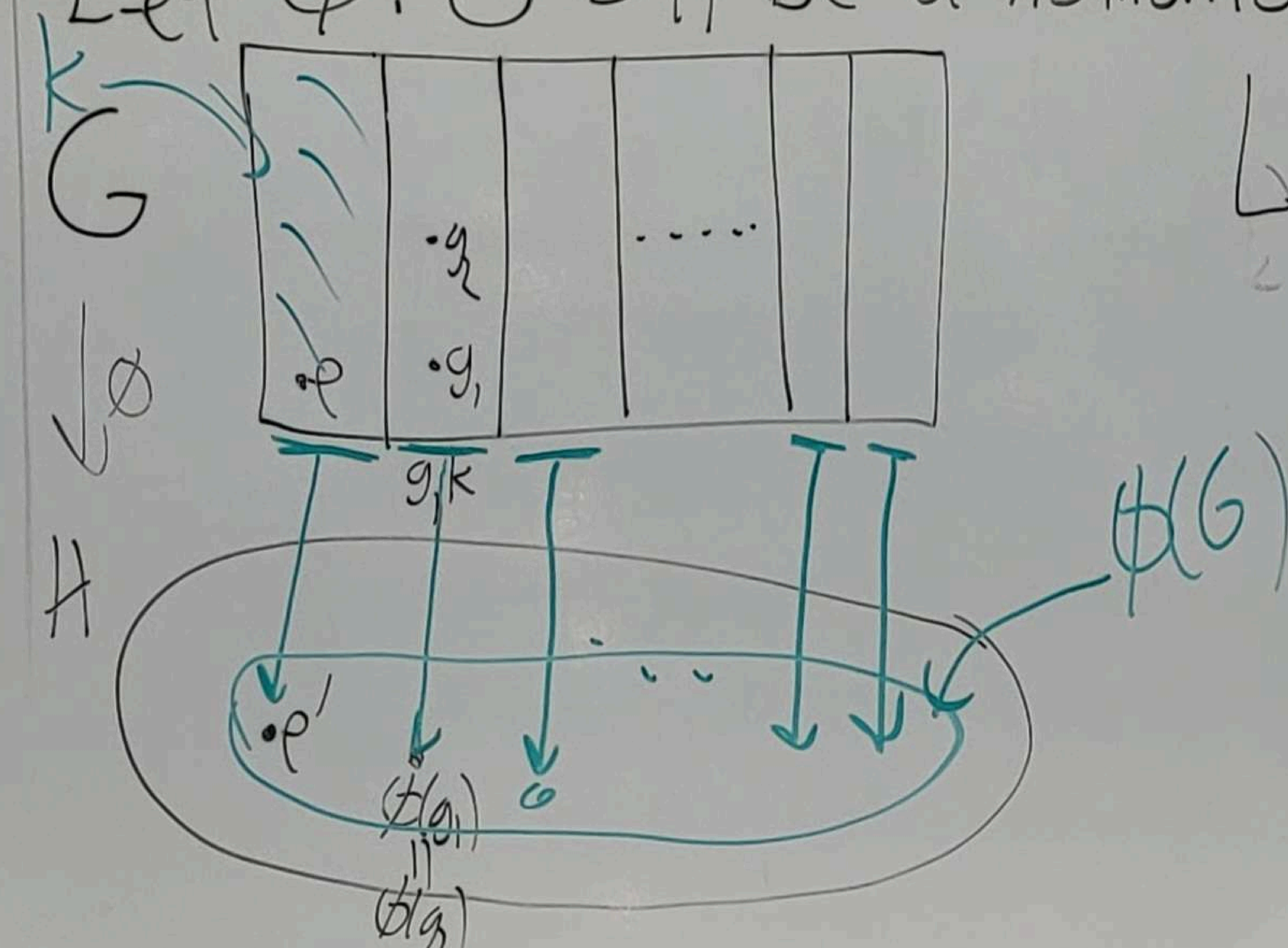
Proof: Let $H = G/N$, and $\phi: G \to G/N$

where $\phi(g) = gN \in G/N$

the canonical homomorphism from G to G/N.

Then $\ker(\phi) = N$.

Let $\phi: G \to H$ be a homomorphism.



Let $k = \ker(\phi) \trianglelefteq G$.

Claim: For $g_1, g_2 \in G$,

$\phi(g_1) = \phi(g_2)$

if and only if

$g_1 K = g_2 K$

Proof ($\Leftarrow$) Let $g_1 K = g_2 K$.

Then $g_1^{-1} g_2 \in K = \ker(\phi)$.

So $\phi(g_1^{-1} g_2) = e'$ ← Identity element of H

But $\phi(g_1^{-1} g_2) = \phi(g_1^{-1}) \phi(g_2)$

$= \phi(g_1)^{-1} \phi(g_2)$

Then $\phi(g_1)^{-1} \phi(g_2) = e'$

So $\phi(g_1) = \phi(g_2)$.

($\Rightarrow$) Suppose $\phi(g_1) = \phi(g_2)$ and show $g_1 K = g_2 K$.

From $\phi(g_1) = \phi(g_2)$ we get

$\phi(g_1^{-1} g_2) = e'$ (reverse above argument)

So $g_1^{-1} g_2 \in K$

and thus

$g_1 K = g_2 K$. $\boxed{\S}$