MATH 3GR3 Midterm #1 Sample Questions Solutions

1. (a) Give the definition of a group.

   Solution: Consult the textbook.

   (b) Let $\mathbb{R}^+ = \{r \in \mathbb{R} : r > 0\}$, the set of all positive real numbers, and for $a, b \in \mathbb{R}^+$, define $a \circ b = \sqrt{ab}$. Is $\mathbb{R}^+$ with the operation $\circ$ a group? Justify your answer.

   Solution: NO. There is no identity element for this operation. To see this, suppose that $r \in \mathbb{R}^+$ has the property that $r \circ b = b$ for all $b \in \mathbb{R}^+$. Then $\sqrt{rb} = b$ and so $rb = b^2$. But then $r = b$ for all $b \in \mathbb{R}^+$, which is impossible.

   In addition, the operation $\circ$ is not associative: for instance, $1 \circ (2 \circ 3) = \sqrt{\sqrt{6}}$, while $(1 \circ 2) \circ 3 = \sqrt{(\sqrt{2})3}$.

2. Here is part of an operation table for an operation on a five element set G whose elements are u, v, w, x, y.

$\cdot$	u	v	w	x	y
u	u	v	w		
v					
w		y	x	v	
x	x			y	
y	y	x			u

   Explain why this table can't be filled in so that the operation it defines satisfies the axioms for a group.

   Solution: We know that every row and column of the Cayley table of a group describes a permutation of the group. So, the third row of the table must be a permutation of $\{u, v, w, x, y\}$. Since the elements y, x, and v already appear in this row, it follows that $wu = u$ or $wy = u$. In the former case, the element u appears twice in the first column of the table, and so this case is not possible. For the latter case, we conclude that u appears twice in the last column, which is also not possible, since u already appears in this column. Since both cases are ruled out, there is no way to complete this table to produce the Cayley table of a group.

3. Produce the Cayley table for $U(12)$, the group of units in $\mathbb{Z}_{12}$.

   Solution: $U(12)$ is the group that consists of the elements $\{1, 5, 7, 11\}$ with operation integer multiplication modulo 12. Thus the Cayley table for this group is:

$\cdot$	1	5	7	11
1	1	5	7	11
5	5	1	11	7
7	7	11	1	5
11	11	7	5	1

4. Consider the following element of S_7: $\mu = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 5 & 1 & 6 & 7 & 2 & 4 & 3 \end{pmatrix}$.

   (a) Write μ as a product of disjoint cycles.

   Solution: $\mu = (152)(3647)$

   (b) What is the order of μ?

   Solution: The order of μ is 12 (12 is the least common multiple of 3 and 4, the orders of the two cycles from part a)).

   (c) Determine if μ is an even or an odd permutation.

   Solution: μ is odd since it is the product of a 3-cycle (which is even) and a 4-cycle (which is odd).

5. Let n be a natural number with $n > 1$ and suppose that G is a cyclic group of order n. Show that if the natural number m divides into n, then G has a subgroup of order m.

   Solution: Suppose that $G = \langle a \rangle$ for some element $a \in G$. Let m be a natural number that divides into n and let $k = n/m$. Let $g = a^k \in G$. Then the order of g is m, since $g^m = (a^{(n/m)})^m = a^n = e$ and if $r < m$ then $g^r = (a^{(n/m)})^r = a^{(nr)/m} \neq e$, since a has order n and $(nr)/m < n$. [Alternatively, we know that the order of a^k is $n/\gcd(n, k) = m$.]

   It follows that the subgroup $H = \langle g \rangle$ has order m.