MATH 4LT3/6LT3 Assignment #1 Solutions

Due: Friday, September 19, by 11:59pm.

Upload your solutions to the Avenue to Learn course website.

Detailed instructions will be provided on the course website.

Please read the following statement on collaboration on homework:

Limited collaboration in planning and thinking through solutions to homework problems is allowed, but no collaboration is allowed in writing up solutions. It is permissible to discuss general aspects of the problem sets with other students in the class, but each person should hand in his/her own copy of the solutions. By general aspects I mean you can say things like, "Did you use a diagonalization argument for question 1?" Anything more detailed than this is not acceptable.

Violation of these rules may be grounds for giving no credit for a homework paper and also for serious disciplinary action.

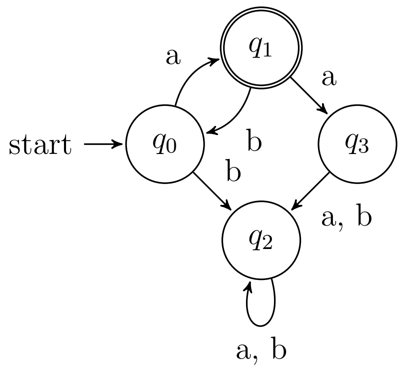
Important note: You may not use any generative Artificial Intelligence system, such as ChatGPT, to assist you in preparing your solutions to the homework problems.

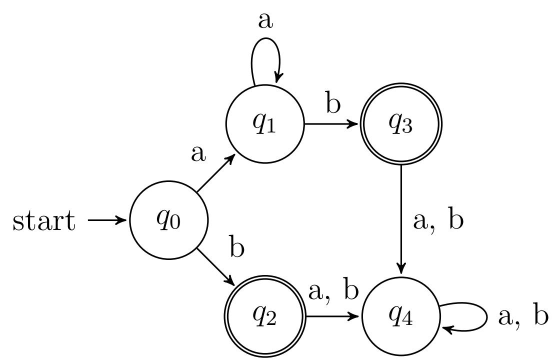
In presenting your solutions, I will be looking for well written, comprehensible answers. Please don't shy away from using complete English sentences to explain your work, and please be careful how you use quantifiers. Every statement you write down should assert something, and should be used somehow to help solve the problem at hand.

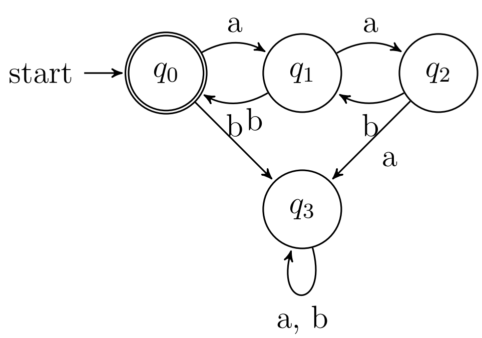
NOTE: You may find it useful to employ the Finite Automata simulator that is linked to from the course homepage (https://www.jflap.org) when working on some of the questions in this assignment. The link will direct you to a simulator that is available as a downloadable java application. You will need to have the appropriate java installation on your computer in order for the simulator to function properly.

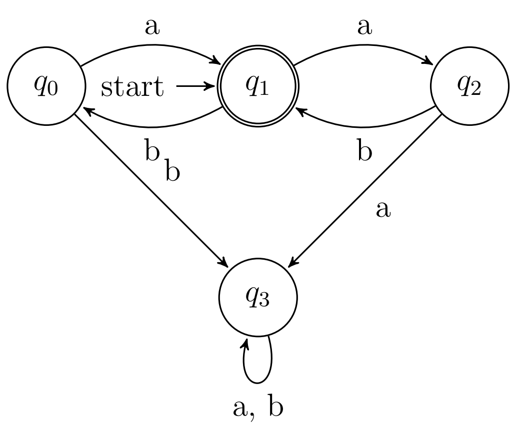
If you use the application for solving any of the homework questions please consider including saved copies of the automata that you construct as separate uploads as part of your Avenue To Learn submission, or submitting print outs of their state diagrams.

1. For each of the following DFA's, provide an informal description of the languages that they accept.









Solution:

a) The language accepted by this DFA is the set of words of the form $(ab)^n a$ for some number $n \geq 0$. This is the same as the set of words of the form $a(ba)^n$, for $n \geq 0$ and can be described by the regular expressions $(((a \cdot b)^*) \cdot a)$ or $(a \cdot ((b \cdot a)^*))$.

b) The language accepted by this DFA is the set of words of the form $a^n b$ for some number $n \geq 0$ and can be described by the reg. exp. $((a^*) \cdot b)$.

c) A word w is accepted by this DFA iff the number of a's in w is equal to the number of b's in w and no initial segment of w contains more b's than a's and never more than 2 a's than b's. The following reg. exp. describes this language: $((ab)^* a (ab)^* b)^*$.

d) The language accepted by this DFA is the set of words over the strings ba and ab, i.e., words from $\{ab, ba\}^*$.

2. Please submit complete solutions to the following exercises from Chen.

(a) Exercise 1.9.1

Solution: ba, aba, bba, baa, bab.

(b) Exercise 1.9.8

Solution: Let B be a language over the alphabet Σ. Let

$$M_B = (Q, \Sigma, s, T, \delta)$$

be the DA with $Q = \Sigma^*$, $s = \epsilon$, $T = B$, and $\delta : Q \times \Sigma \to Q$ defined by $\delta(\sigma, a) = \sigma \cdot a$, for any string $\sigma \in Q = \Sigma^*$, and $a \in \Sigma$. We claim that $L(M_B) = B$. This follows from

Claim: For $\sigma, \beta \in \Sigma^*$, $[\beta, \sigma] \vdash^n_{M_B} [\beta \cdot \sigma, \epsilon]$, where $n = |\sigma|$.

Proof. We prove this by induction on $|\sigma|$. For $|\sigma| = 0$, we have that $\sigma = \epsilon$. It is clear that the claim holds in this case. Suppose that it holds for all strings of length n for some $n \geq 0$ and let $|\sigma| = n + 1$. Then $\sigma = a \cdot \alpha$ for some $\alpha \in \Sigma^*$ and $a \in \Sigma$, with $|\alpha| = n$.

By induction,

$$[\beta \cdot a, \alpha] \vdash^n_{M_B} [\beta \cdot a \cdot \alpha, \epsilon] = [\beta \cdot \sigma, \epsilon].$$

Since $\delta(\beta, a) = \beta \cdot a$, it follows that

$$[\beta, \sigma] = [\beta, a \cdot \alpha] \vdash_{M_B} [\beta \cdot a, \alpha]$$

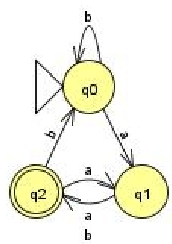
and so $[\beta, \sigma] \vdash_{M_B}^{n+1} [\beta \cdot \alpha, \epsilon]$, as claimed. □

Using the definition of acceptance, we have that a string σ is accepted by M_B if and only if $[\epsilon, \sigma] \vdash_{M_B}^{*} [q, \epsilon]$ for some state $q \in T = B$. But by the claim, q must be equal to σ and so σ is accepted by M_B if and only if $\sigma \in B$. Thus $L(M_B) = B$.

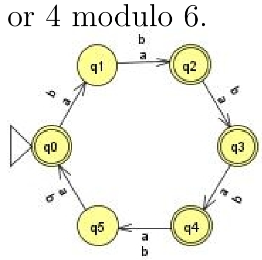
(c) Exercise 1.9.13, #2, 4, 7, 9, 14

Solution:

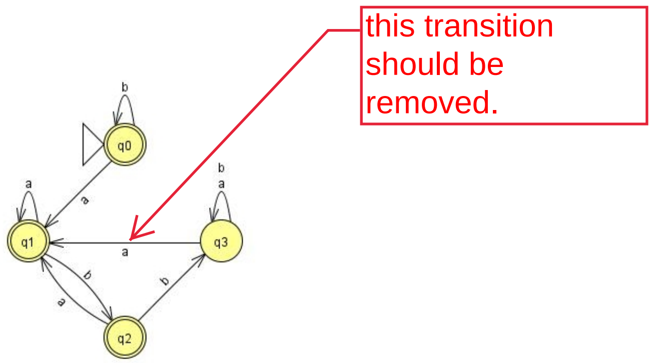
#2



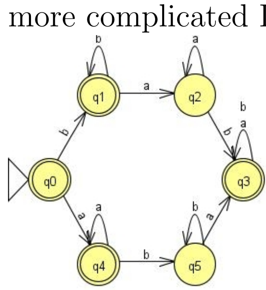
#4 The condition is equivalent to $|x|$ being congruent to 0, 2, 3, or 4 modulo 6.



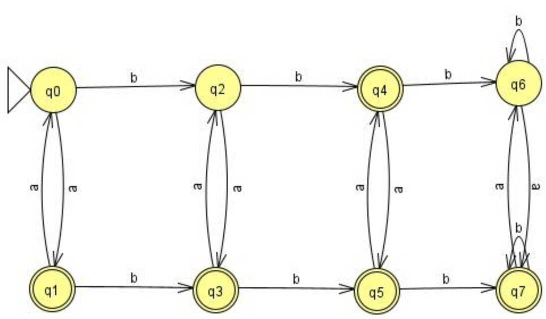
#7 First consider the complement of this language. This is the set of strings that has *abb* as a substring. Then the complementation construction from Section 1.2 can be used to build the required DFA.



#9 Again, it might be useful to first consider the complement of this language. This is the set of strings that contain ab as a substring, but not ba, along with the set of strings that contain ba as a substring, but not ab. In the first case, these are just the strings $a^n b^m$ for some n, $m \geq 1$. In the latter case, these are the strings $b^n a^m$ for some n, $m \geq 1$. Both of these sets of strings can be recognized by simple DFAs. The union of these sets of strings can be recognized by a slightly more complicated DFA.



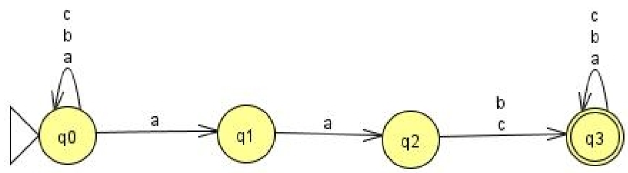
#14 This uses the union construction found in the proof of Theorem 1.2.4.



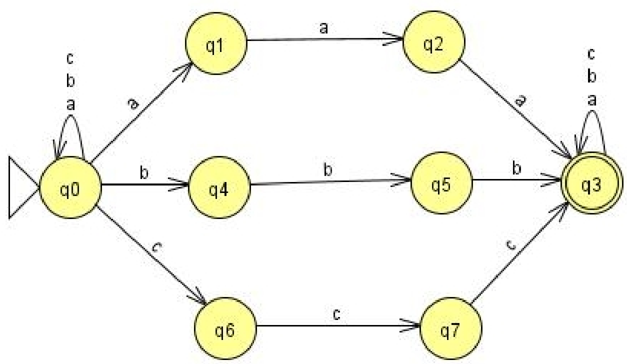
(d) Exercise 1.9.15, #2, 5, 7

Solution:

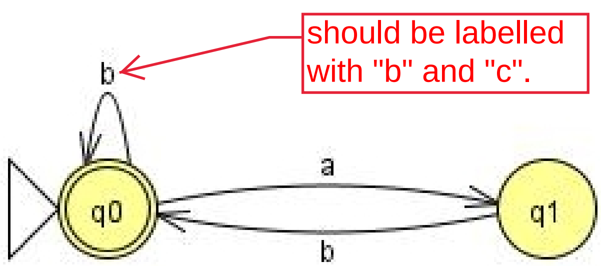
#2



#5



#7



(e) Exercise 1.9.20

Solution: This DFA can be obtained by applying the product construction to the two 2-states DFAs that accept those strings that have an even number of occurrences of a and those strings that have an even number of occurrences of b respectively.

(f) Exercise 1.9.23

Solution: We should first consider the trivial case where $B = \emptyset$. Then $P = \emptyset$ too and so there is nothing to prove. Now suppose that B is nonempty and let $M = (Q, \Sigma, q_0, T, \delta)$ be a DFA with $L(M) = B$. Note that since B is nonempty then $\epsilon \in P$. To show that P is regular, we can modify M by enlarging the set of accept states T of M to T' to produce a DFA M' with $L(M') = P$. We can regard the diagram for M as a directed graph whose edges are labelled by symbols from Σ. A path from state q to state r is a sequence of edges that connect q to r. If we define T' to be the set of states q from Q for which there is a path to a state $r \in T$ then it can be shown that the language of the DFA $M' = (Q, \Sigma, q_0, T', \delta)$ is P.

More precisely, define

$$T' = \{q \in Q \mid [q, w] \vdash_M^* [r, \epsilon] \text{ for some } r \in T \text{ and } w \in \Sigma^*\}.$$

So T' consists of those states for which there is a path to some accept state of M. We note that since B is nonempty then for some (any) string $w \in B$, $[s, w] \vdash_M^* [t, \epsilon]$ for some $t \in T$ and so $s \in T'$.

We claim that for all strings x, $x \in P$ if and only if M' accepts x. If $x \in P$ then $xv \in B = L(M)$ for some string v. This means that $[s, xv] \vdash_M^* [t, \epsilon]$ for some $t \in T$. Let $q \in Q$ be the unique state with

$$[s, xv] \vdash_M^* [q, v] \vdash_M^* [t, \epsilon].$$

Then by definition, $q \in T'$ and it can be seen that $[s, x] \vdash_M^* [q, \epsilon]$ Since M and M' have the same start state and transition function, we also have $[s, x] \vdash_{M'}^* [q, \epsilon]$, showing that $x \in L(M')$.

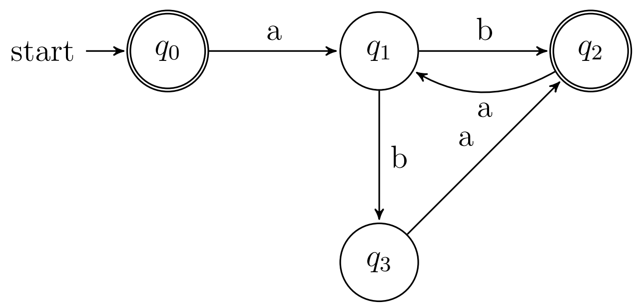
Conversely, suppose that $x \in L(M')$. Then for some $q \in T'$, $[s, x] \vdash_{M'}^* [q, \epsilon]$. As noted earlier, this implies that $[s, x] \vdash_M^* [q, \epsilon]$ as well. Since $q \in T'$ then there is some $v \in \Sigma^*$ and $t \in T$ with $[q, v] \vdash_M^* [t, \epsilon]$. It follows that

$$[s, xv] \vdash_M^* [q, v] \vdash_M^* [t, \epsilon],$$

showing that $xv \in L(M) = B$. But this also shows that $x \in P$, as claimed.

Note that in the above, we claimed without proof that for any strings x and v and states q and r, $[q, x] \vdash_M^* [r, \epsilon]$ if and only if $[q, xv] \vdash_M^* [r, v]$. This can be proved by induction on $|x|$, with the case $|x| = \epsilon$ being trivial. We leave the details of the rest of the proof to the reader.

3. Consider the following NFA:



   (a) Determine which of the following strings are accepted by this NFA: ϵ, ab, $abab$, aba, $abaa$.

   Solution: Part (a): All words in the list, except for $abaa$ are accepted by the NFA.

   Note that part (b) of this question will appear on Assignment #2.

The following (multi-part) question is for students enrolled in MATH 6LT3. Students in MATH 4LT3 can treat it as a bonus question.

Let $M = (Q, \Sigma, q_0, F, \delta)$ be a DFA.

1. For σ, $\tau \in \Sigma^*$, define $\sigma \equiv \tau$ if for all $\omega \in \Sigma^*$, $\sigma\omega \in L(M)$ if and only if $\tau\omega \in L(M)$. Show that $\equiv$ is an equivalence relation on Σ^*.

   Solution: $\equiv$ is clearly reflexive and symmetric. For transitivity, suppose that $\sigma \equiv \tau \equiv \gamma$ and let $w \in \Sigma^*$. Then $\sigma w \in L(M)$ if and only if $\tau w \in L(M)$ if and only if $\gamma w \in L(M)$. This establishes that $\sigma \equiv \gamma$ and hence the transitivity of $\equiv$.

2. For σ, $\tau \in \Sigma^*$, define $\sigma \sim \tau$ if $\delta(q_0, \sigma) = \delta(q_0, \tau)$. Show that $\sim$ is an equivalence relation on Σ^* and that $\sim$ is a refinement of $\equiv$. Argue that the number of equivalence classes of $\equiv$ is at most $|Q|$, the number of states of M. Note that we are regarding δ as an extended transition function.

   Solution: $\sim$ is an equivalence relation since it is the kernel of some function, namely $f(x) = \delta(q_0, x)$. In general, for any function $f(x)$, the relation $\{(a, b) \mid f(a) = f(b)\}$ is always an equivalence relation on the domain of f.

   Suppose that $\sigma \sim \tau$ and let $w \in \Sigma^*$ with $\sigma w \in L(M)$. This means that $\delta(q_0, \sigma w) = q$ for some accepting state q of M. But then

   $$\delta(q_0, \tau w) = \delta(\delta(q_0, \tau), w) = \delta(\delta(q_0, \sigma), w) = \delta(q_0, \sigma w) = q,$$

   establishing that $\tau w \in L(M)$ too. From this it follows that $\sigma \equiv \tau$, as required. Since the number of $\sim$ classes is bounded by the number of states of M and since the number of $\equiv$ classes is at most the number of $\sim$ classes, it follows that the number of $\equiv$ classes is bounded by $|Q|$.

3. For $\sigma \in \Sigma^*$, let $[\sigma]$ denote the equivalence class of strings that are $\equiv$-related to σ. Define M' to be the DFA $(Q', \Sigma, \delta', [\epsilon], F')$, where

   (a) $Q' = \{[\sigma] \mid \sigma \in \Sigma^*\}$,

   (b) $F' = \{[\sigma] \mid \sigma \in L(M)\}$, and

   (c) For $\sigma \in \Sigma^*$ and $a \in \Sigma$, $\delta'([\sigma], a) = [\sigma \cdot a]$.

   Show that δ' is a well-defined function. Then show that $L(M') = L(M)$.

   Solution: To show that δ' is well-defined, we need to show that if σ, $\tau \in \Sigma^*$ with $\sigma \equiv \tau$ then for every $a \in \Sigma$, $[\sigma \cdot a] = [\tau \cdot a]$, i.e., that $\sigma \cdot a \equiv \tau \cdot a$. This follows from the definition, since for any $w \in \Sigma^*$, if $(\sigma \cdot a) \cdot w \in L(M)$ then $\sigma \cdot w' \in L(M)$, where $w' = a \cdot w$. But then, since $\sigma \equiv \tau$, we have $\tau \cdot w' \in L(M)$ as well, which is the same as $(\tau \cdot a) \cdot w \in L(M)$. From this it follows that $\sigma \cdot a \equiv \tau \cdot a$, as required.

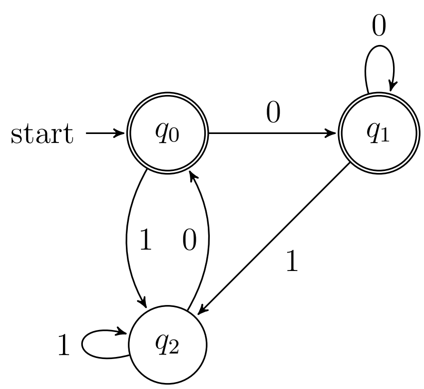
   For the second part, first note that for all $w \in \Sigma^*$, $\delta'([\epsilon], [w]) = [w]$. This can be proved by induction on $|w|$. From this we conclude that $w \in L(M')$ if and only if $[w] \in F'$, i.e., $w \equiv \sigma$ for some $\sigma \in L(M)$.

But then we have $w \in L(M)$ as well. Conversely, if $w \in L(M)$, then $[w] \in F'$. Since $\delta'([\epsilon], w) = [w]$ it follows that $w \in L(M')$.

4. Explain why, amongst all of the DFA's that accept the language $L(M)$, M' has the fewest number of states.

   Solution: By construction, the number of states of M' is not dependent on M, but rather on the language $L = L(M)$. Also, as shown in part 2), the number of states of M' is at most the number of states of any DFA M that accepts L. It follows that the number of states of M' is the smallest amongst those DFA's that accept L.

5. For the following DFA M, construct the DFA M' using the above definition.



   Solution: We first need to construct the $\equiv$ classes for $L(M)$. Note that $L(M)$ is the set of all words over $\{0, 1\}$ that do not end in 1. From this it can be seen that there are exactly two $\equiv$ classes: $[\epsilon] = \{w \in \{0,1\}^* \mid w = \epsilon \text{ or } w \text{ ends in } 0\}$ and $[1] = \{w \in \{0,1\}^* \mid w \text{ ends in } 1\}$.

   So M' has two states, $[\epsilon]$ and $[1]$, with $[\epsilon]$ the initial state and the only accepting state. The transition function for M' maps $([\epsilon], 0)$ to $[\epsilon]$, $([\epsilon], 1)$ to $[1]$, $([1], 0)$ to $[\epsilon]$, and $([1], 1)$ to $[1]$.