MATH 4LT3/6LT3 Assignment #2 Solutions
Due: Friday, October 6, 11:59pm
Upload your solutions to the Avenue to Learn course website.
Detailed instructions will be provided on the course website.

1. In the text, the Kuratowski definition of an ordered pair is given along with a proof that it satisfies the ordered pair properties (OP1) and (OP2).

   (a) Show that the following construction also satisfies these properties, where $0 = \emptyset$, and $1 = \{\emptyset\}$:

   $$(x, y) = \{\{x, 0\}, \{y, 1\}\}.$$

   Solution: (OP1): Suppose that $(x, y) = (x', y')$. Then $\{\{x, 0\}, \{y, 1\}\} = \{\{x', 0\}, \{y', 1\}\}$. Either $\{x, 0\} = \{x', 0\}$ and $\{y, 1\} = \{y', 1\}$ or $\{x, 0\} = \{y', 1\}$ and $\{y, 1\} = \{x', 0\}$. In the former case, either $x = x'$ or $x = 0$ and $x' = x$. In any case we conclude that $x = x'$. Similarly, from $\{y, 1\} = \{y', 1\}$ we conclude that $y = y'$, as desired. In the latter case, from $\{x, 0\} = \{y', 1\}$ we get that since $0 \neq 1$, then $x = 1$ (since $1 \in \{x, 0\}$) and $y' = 0$ (since $0 \in \{y', 1\}$), and from $\{y, 1\} = \{x', 0\}$ we get that $y = 0$ and $x' = 1$. Putting this together, we conclude that $x = 1 = x'$ and $y = 0 = y'$.
   (OP2): We see that for sets x and y, the ordered pair (x, y) is a subset of the set $\mathcal{P}(\{x, y, 0, 1\})$ and so is a member of $\mathcal{P}(\mathcal{P}(\{x, y, 0, 1\}))$. If $x \in A$ and $y \in B$, then (x, y) is a subset of $\mathcal{P}(A \cup B \cup \{0, 1\})$. So, $A \times B$ is the set

   $$\{z \in C \mid z = \{\{x, 0\}, \{y, 1\}\} \text{ for some } x \in A \text{ and } y \in B\},$$

   where $C = \mathcal{P}(\mathcal{P}(A \cup B \cup \{0, 1\}))$. It follows that $A \times B$ is a set, by the Separation Axiom.

   (b) Determine if the following construction satisfies the ordered pair properties:

   $$(x, y) = \{x, \{x, y\}\}.$$

   Solution: Without additional axioms, it can't be deduced that (x, y) satisfies (OP1). Without the Axiom of Foundation (we'll see it later), we could have distinct sets x and y with $x = \{y\}$ and $y = \{x\}$. For such sets, it can be seen that $(x, x) = (y, y)$.

2. Recall the definition of a cardinal assignment from the lectures. Given such an assignment (weak or strong), show that if κ, λ, and μ are cardinals, then $(\kappa \cdot \lambda)^\mu =_c \kappa^\mu \cdot \lambda^\mu$ and $(\kappa^\lambda)^\mu =_c \kappa^{\lambda\mu}$.

   Solution: Let K, L, and M be disjoint sets such that $\kappa = |K|$, $\lambda = |L|$, and $\mu = |M|$. Then $\kappa \cdot \lambda =_c K \times L$ and $(\kappa \cdot \lambda)^\mu =_c (M \to K \times L)$. $\kappa^\mu \cdot \lambda^\mu =_c (M \to K) \times (M \to L)$. Since the sets $(M \to K \times L)$ and $(M \to K) \times (M \to L)$ are equinumerous (a function f from M to $K \times L$ is uniquely determined by the functions $f_1 = \pi_1 \circ f$ in $(M \to K)$ and $f_2 = \pi_2 \circ f$ in $(M \to L)$, where π_1 is the projection map from $K \times L$ to K and π_1 is the projection map from $K \times L$ to L). So

   $$(\kappa \cdot \lambda)^\mu =_c (M \to K \times L) =_c (M \to K) \times (M \to L) =_c \kappa^\mu \cdot \lambda^\mu.$$

   The other equality follows from the fact that $(M \to (L \to K)) =_c ((M \times L) \to K)$.

3. With $\mathfrak{c}$ the cardinality of the continuum (technically, $|\mathcal{P}(\mathbb{N})|$), show that $\mathfrak{c}^\mathfrak{c} =_c 2^\mathfrak{c}$. You might consider using the results from the previous problem, and also first establishing that $\aleph_0 \cdot \mathfrak{c} =_c \mathfrak{c}$.

   Solution: First show that $\aleph_0 \cdot \mathfrak{c} =_c \mathfrak{c}$. This amounts to showing that

   $$\mathbb{N} \times \mathcal{P}(\mathbb{N}) =_c \mathcal{P}(\mathbb{N}).$$

   Clearly $\mathcal{P}(\mathbb{N}) \leq_c \mathbb{N} \times \mathcal{P}(\mathbb{N})$, so by the Schröder-Bernstein Theorem, it suffices to show that $\mathbb{N} \times \mathcal{P}(\mathbb{N}) \leq_c \mathcal{P}(\mathbb{N})$. The following is one (of many) injections from $\mathbb{N} \times \mathcal{P}(\mathbb{N})$ to $\mathcal{P}(\mathbb{N})$: $(n, S) \mapsto S'$, where $m \in S'$ if and only if $m = 2n + 1$ or $m = 2s$ for some $s \in S$. The only odd number in S' is $2n + 1$, so n is uniquely determined by S', and the elements of S can be obtained by dividing in two all even members of S'.

   To show that $\mathfrak{c}^\mathfrak{c} =_c 2^\mathfrak{c}$, consider that

   $$\mathfrak{c}^\mathfrak{c} =_c (2^{\aleph_0})^\mathfrak{c} =_c 2^{\aleph_0 \cdot \mathfrak{c}} =_c 2^\mathfrak{c}.$$

   The second equality follows from the previous question, and the third from the first part of this question.

4. For $k \in \mathbb{N}$, let the function $f_k : \mathbb{N} \to \mathbb{N}$ be defined by

   $$f_k(n) = k^{(k^{(k^{\cdot^{\cdot^{\cdot^{k)}}}})}},$$

where k appears n-times. Show that there exists a function f with domain $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$ such that $f(k, n) = f_k(n)$. You should consider using the recursion with parameters theorem for this. Next, show that the function

$$e(n) = n^{(n^{(n^{\cdot^{\cdot^{\cdot^{n)}}}})}},$$

where n appears n-times, is a member of the set $(\mathbb{N} \to \mathbb{N})$.

Solution: To set up an application of the recursion with parameters theorem, define $g : \mathbb{N} \to \mathbb{N}$ by $g(k) = 1$, and $h : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ by $h(m, n) = n^m$. Then by the theorem, there is a unique function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ that satisfies $f(k, 0) = g(k) = 1 = f_k(0)$, and $f(k, Sn) = h(f(k, n), k) = k^{f(k,n)}$.

It can be seen (inductively) that $f(k, n) = f_k(n)$ for all k and n. Finally, $e(n) = f(n, n)$ for all n.

5. Now that we have constructed the natural numbers, i.e., the structured set $(\mathbb{N}, +, \times, \leq, 0, 1)$, show that the integers $(\mathbb{Z}, +, \times, \leq, 0, 1)$ can be faithfully represented as a structured set within our set theoretic universe $\mathcal{W}$. You will need to describe a construction of the integers, along with the operations of $+$, $\times$, the relation $\leq$, and the elements 0 and 1 from the natural numbers, that can be carried out using, indirectly, the Axioms.

   Solution: We can represent the elements of $\mathbb{Z}$ with the set $\mathbb{Z}' = \{0, 1\} \times \mathbb{N}$, with a non-negative integer n corresponding to the pair $(1, n)$ and with a negative integer n corresponding to the pair $(0, -n)$. This is clearly a bijection from $\mathbb{Z}$ to $\mathbb{Z}'$. Using definition by cases (which corresponds to taking unions of functions, when viewed as sets), we can define $+$ by:

$$(i, n) + (j, m) = \begin{cases} (i, n+m) & \text{if } i = j \\ (1, m-n) & \text{if } i = 0, j = 1, m \geq n \\ (0, n-m) & \text{if } i = 0, j = 1, m < n \\ (1, n-m) & \text{if } i = 1, j = 0, n \geq m \\ (0, m-n) & \text{if } i = 0, j = 1, n < m \end{cases}$$

The operation $\times$ on $\mathbb{Z}'$ can be defined similarly. The element 0 corresponds to $(0,0)$ and 1 corresponds to $(1,1)$. The relation $\leq$ on $\mathbb{Z}'$ is defined by:

> $(i,m) \leq (j,n)$ if and only if $i = j = 1$ and $m \leq n$ or $i = 0$ and $j = 1$, or $i = j = 0$ and $m \geq n$.

6. Let $(P, \leq)$ and $(Q, \preceq)$ be linearly ordered sets.

   (a) Define their sum to be the order over the disjoint union of P and Q such that elements of P are less than all of the elements of Q, and elements within P or Q are ordered according to $\leq$ or $\preceq$ respectively.

   Show that the sum of $(P, \leq)$ and $(Q, \preceq)$ is a linear order. If they are both well orders, is their sum?

   Solution: We may assume that P and Q are disjoint sets. The sum of the two orders is the structured set $(P \cup Q, \ll)$, where for $p, p' \in P$, $p \ll p'$ if and only if $p \leq p'$, for $q, q' \in Q$, $q \ll q'$ if and only if $q \preceq q'$, and for $p \in P$ and $q \in Q$, $p \ll q$. More concisely, as a set of ordered pairs over $P \cup Q$, $\ll = \leq \cup \preceq \cup P \times Q$.

   The relation $\ll$ is reflexive since both $\leq$ and $\preceq$ are. For transitivity, if a, b, $c \in P \cup Q$ with $a \ll b \ll c$ then if $c \in P$ it follows that a and $b \in P$ and so $a \leq b \leq c$ and thus $a \leq c$, so $a \ll c$. Similarly if $a \in Q$, it can be seen that $a \ll c$. The remaining case is when $a \in P$ and $c \in Q$. Then it follows that $a \ll c$, as required.

   If $a \ll b$ and $b \ll a$, then both must lie in P or lie in Q. Then the antisymmetries of $\leq$ and $\preceq$ implies that $a = b$. So $\ll$ is a partially ordered set (we haven't used linearity of $\leq$ or $\preceq$ yet). To see that $\ll$ is linear, let a, $b \in P \cup Q$. If both lie in P or Q then they are comparable via $\leq$ or $\preceq$ respectively, which means that they are $\ll$-comparable. In the remaining case, one lies in P and the other in Q and so are also $\ll$-comparable. Thus $\ll$ is a linear order.

   Finally, if both are well orders, then so is $\ll$: Let $W \subseteq P \cup Q$ be nonempty and let $W_P = W \cap P$ and $W_Q = W \cap Q$. If W_P is nonempty, then the $\leq$-least element of W_P will be the $\ll$-least element of W. Otherwise, $W = W_Q$ and the $\preceq$-least member of W will be the $\ll$-least member of W.

(b) Define the product of these linearly ordered sets to be the order $\sqsubseteq$ on the set $P \times Q$ such that $(p, q) \sqsubseteq (p', q')$ if and only if $(q \prec q')$ or $(q = q'$ and $p \leq p')$.

Show that the product of $(P, \leq)$ and $(Q, \preceq)$ is a linear order. If they are both well orders, is their product?

Solution: To see that $\sqsubseteq$ is a linear order on $P \times Q$ is routine and left to the reader. To see that it is a well order, let $W \subseteq P \times Q$ be nonempty. Let $W_Q = \{q \in Q \mid (p, q) \in W \text{ for some } p \in P\}$. Since W is nonempty, then so is W_Q, and so has a $\preceq$-least element q. Since $q \in W_Q$, then the set $W_P = \{p \in P \mid (p, q) \in W\}$ is nonempty and so has a $\leq$-least element p. Then $(p, q) \in W$ and is the $\sqsubseteq$-least member of W. Thus $\sqsubseteq$ is a well order.