new due date: Monday, 13 November.

MATH 4LT3/6LT3 Assignment #4

Due: Friday, 10 November, by 11:59pm

Unless otherwise stated, in your solutions you may use the Axiom of Choice, or any equivalent statement that has been discussed in the lectures

1. Let $(U, \leq)$ be a well order. Show that $(U, \leq) =_o (\{\mathrm{Seg}_{\leq}(u) \mid u \in U\}, \subseteq)$.

2. Let G be an infinite non-abelian group. Show that it has a maximal abelian subgroup, i.e., that there is some subgroup H of G such that H is abelian and if K is an abelian subgroup of G such that $H \subseteq K$, then $H = K$. Note that every group has at least one abelian subgroup, $\{e\}$.

3. Consider the following principle:

   > For any sets A and B and surjective function $f : A \twoheadrightarrow B$, there is a one-to-one function $g : B \rightarrowtail A$ such that $f(g(b)) = b$ for all $b \in B$.

   Show that this principle is equivalent to the Axiom of Choice.

   The following is a formally weaker principle than the one above, and so can be deduced from AC (you don't need to show this):

   > For any sets A and B and surjective function $f : A \twoheadrightarrow B$, there is a one-to-one function $g : B \rightarrowtail A$.

   One of the oldest open problems in Set Theory is to determine if this principle implies AC.

4. Related to the previous question, use AC to show that if A and B are sets such that there are surjections $f : A \twoheadrightarrow B$ and $g : B \twoheadrightarrow A$, then $A =_c B$. It is not known if the converse holds, namely, if this principle implies AC.

5. Consider the following statement:

   > Let A and B be non-empty sets and $f : A \rightarrowtail B$ be an injective function from A to B. Then there is an onto function $g : B \twoheadrightarrow A$ such that $g(f(a)) = a$ for all $a \in A$.

(a) Prove that this statement.

(b) Is this statement equivalent to the Axiom of Choice?

6. Let A and B be sets and let $(A \hookrightarrow B)$ denote the set of partial functions from A to B that are one-to-one. (The text uses a slightly different notation.) So $f \in (A \hookrightarrow B)$ if there is some subset $C \subseteq A$ such that $f \in (C \rightarrowtail B)$. The set $(A \hookrightarrow B)$ can naturally be ordered by inclusion: $f \leq g$ if and only if for all $a \in \text{Domain}(f)$, $a \in \text{Domain}(g)$ and $g(a) = f(a)$. (So considered as sets of ordered pairs, $f \subseteq g$.) It is not hard to see that this ordering is a partial ordering on $(A \hookrightarrow B)$.

(a) Show that if $C \subseteq (A \hookrightarrow B)$ is a chain in this poset, then $\cup C$ is in $(A \hookrightarrow B)$ and is an upper bound of C in the poset.

(b) Show that this poset has a maximal element.

(c) Let $f \in (A \hookrightarrow B)$ be maximal in this poset. Show that either $\text{Domain}(f) = A$, and so f is an injection from A to B, or $\text{Image}(f) = B$, and so $B =_c \text{Domain}(f)$.

(d) Conclude that for all sets A and B, either $A \leq_c B$ or $B \leq_c A$.