MATH 4LT3/6LT3 Assignment #4 Solutions
Due: Friday, 10 November, by 11:59pm

Unless otherwise stated, in your solutions you may use the Axiom of Choice, or any equivalent statement that has been discussed in the lectures

1. Let $(U, \leq)$ be a well order. Show that $(U, \leq) =_o (\{\mathrm{Seg}_{\leq}(u) \mid u \in U\}, \subseteq)$.

   Solution: Define $f : U \to \{\mathrm{Seg}_{\leq}(u) \mid u \in U\}$ by $f(u) = \mathrm{Seg}_{\leq}(u)$ for $u \in U$. The function f is seen to be an order-preserving bijection since for v, $u \in U$, $\mathrm{Seg}_{\leq}(u) \subset \mathrm{Seg}_{\leq}(v)$ if and only if $u < v$. Thus $(U, \leq) =_o (\{\mathrm{Seg}_{\leq}(u) \mid u \in U\}, \subseteq)$.

2. Let G be an infinite non-abelian group. Show that it has a maximal abelian subgroup, i.e., that there is some subgroup H of G such that H is abelian and if K is an abelian subgroup of G such that $H \subseteq K$, then $H = K$. Note that every group has at least one abelian subgroup, $\{e\}$.

   Solution: We can use Zorn's Lemma to establish this. Let P be the set of all proper abelian subgroups H of G, ordered by inclusion. Then P is a poset and it is non-empty, since the trivial subgroup $\{e\}$ is abelian and so belongs to P.

   Let C be a chain in P. To see that C has an upper bound in P, consider the subset $H = \bigcup C$, the union of all of the subgroups $K \in C$. Clearly every member $K \in C$ is a subset of H, so to show that H is an upper bound for C, we need to show that it belongs to P, i.e., that it is an abelian subgroup of G.

   In any group G, if C is a chain of (abelian) subgroups of G, then its union H will also be a (an abelian) subgroup. This is because

   - $e \in H$ (since e belongs to every member of C),
   - if g_1, $g_1 \in H$, then there will be some K_1, $K_2 \in C$ with $g_1 \in K_1$ and $g_2 \in K_2$. Since C is a chain, then either $K_1 \subseteq K_2$ or $K_2 \subseteq K_1$. In the first case, g_1, $g_2 \in K_2$ and so $g_1 g_2 \in K_2 \subseteq H$. In the latter case, we get the same conclusion, showing that H is closed under the group operation.

If the K_i are abelian, then $g_1 g_2 = g_2 g_1$, since both g_1 and g_2 will belong to one of the K_i. Thus, in this case, the operation of G, restricted to H is commutative.

- if $g \in H$, then there will be some $K \in C$ with $g \in K$. Then $g^{-1} \in K \subseteq H$, showing that H is closed under taking inverses.

This shows that H belongs to P and is an upper bound for C. So by Zorn's Lemma, P has a maximal element, M. Then M is a maximal abelian subgroup of G, since it can't be properly contained in any other abelian subgroup of G.

3. Consider the following principle:

> For any sets A and B and surjective function $f : A \twoheadrightarrow B$, there is a one-to-one function $g : B \rightarrowtail A$ such that $f(g(b)) = b$ for all $b \in B$.

Show that this principle is equivalent to the Axiom of Choice.

The following is a formally weaker principle than the one above, and so can be deduced from AC (you don't need to show this):

> For any sets A and B and surjective function $f : A \twoheadrightarrow B$, there is a one-to-one function $g : B \rightarrowtail A$.

One of the oldest open problems in Set Theory is to determine if this principle implies AC.

Solution: Assume AC and show that this principle holds: define $P \subseteq B \times A$ by $(b, a) \in P$ if and only if $f(a) = b$. Then since f is onto, it follows that for each $b \in B$, there is some $a \in A$ such that $(b, a) \in P$. Then by the axiom of choice, there is some function $g : B \to A$ such that for $b \in B$, $(b, g(b)) \in P$. This means that $f(g(b)) = b$ for all $b \in B$. It is not hard to see that this function g is one-to-one.

Alternatively, let ϵ_A be a choice function for the set A. For $b \in B$, let $S_b = \{a \in A \mid f(a) = b\}$. Since f is onto, then S_b is non-empty for each $b \in B$. Define $g : B \to A$ by $g(b) = \epsilon_A(S_b)$. Then $f(g(b)) = f(\epsilon_A(S_b)) = b$, as required.

Conversely, suppose that this principle holds. It is simpler to show that the following condition, which has been shown to be equivalent to AC, holds. Let $\mathcal{E}$ be a set of disjoint non-empty sets. We want to show that there is a subset $S \subseteq \bigcup \mathcal{E}$ such that for each $X \in \mathcal{E}$, $S \cap X$ is a singleton, i.e., S is a selection set for $\mathcal{E}$.

Let $A = \bigcup \mathcal{E}$ and define $f : A \to \mathcal{E}$ by $f(a) = X$, where X is the unique member of $\mathcal{E}$ with $a \in X$. Since the members of $\mathcal{E}$ are disjoint, then there will be exactly one $X \in \mathcal{E}$ with $a \in X$, so f is well defined (and is a set, since it is a subset of $A \times \mathcal{E}$). It is also the case that f is surjective, since each member of $\mathcal{E}$ is non-empty. So, by the given principle, there is some function $g : \mathcal{E} \rightarrowtail A$ such that $f(g(X)) = X$ for all $X \in \mathcal{E}$. Let $S = g[\mathcal{E}]$, the image of g. Then S is a selection set for $\mathcal{E}$, since for $X \in \mathcal{E}$, $g(X) \in X$. Since the members of $\mathcal{E}$ are disjoint, then there will be exactly one member of S that belongs to X, namely, $g(X)$.

4. Related to the previous question, use AC to show that if A and B are sets such that there are surjections $f : A \twoheadrightarrow B$ and $g : B \twoheadrightarrow A$, then $A =_c B$. It is not known if the converse holds, namely, if this principle implies AC.

   Solution: This follows from the Schröder-Bernstein Theorem, since by applying the result from the previous question to $f : A \twoheadrightarrow B$, we obtain an injection from B to A, showing that $B \leq_c A$. Similarly, from $g : B \twoheadrightarrow A$, we can conclude that $A \leq_c B$, so $A =_c B$.

   This result can be viewed as a surjection version of the Schröder-Bernstein Theorem.

5. Consider the following statement:

   > Let A and B be non-empty sets and $f : A \rightarrowtail B$ be an injective function from A to B. Then there is an onto function $g : B \twoheadrightarrow A$ such that $g(f(a)) = a$ for all $a \in A$.

   (a) Prove this statement.

   Solution: Given A, B, and f as in the statement of this problem, let $a_0 \in A$ and define $g : B \to A$ by

$$g(b) = \begin{cases} f^{-1}(b) & \text{if } b \in f[A] \\ a_0 & \text{otherwise} \end{cases}.$$

This function is well defined, since f is injective, and so for each $b \in f[A]$, there is a unique $a \in A$ with $f(a) = b$ (so $a = f^{-1}(b)$).

It is not hard to see that g is surjective, since for each $a \in A$, there is some $b \in B$ with $f(a) = b$ (so $g(b) = a$), and that for $a \in A$, $g(f(a)) = a$.

(b) Is this statement equivalent to the Axiom of Choice?

Solution: Note that in the solution to part (a), we did not need to make use of any choice function or principle. So this principle follows from the first six Zermelo axioms, without the use of AC. If this principle were equivalent to AC, then it would follow that AC could be proved from the first six Zermelo axioms, which has been shown to not be possible (unless these axioms are inconsistent).

6. Let A and B be sets and let $(A \hookrightarrow B)$ denote the set of partial functions from A to B that are one-to-one. (The text uses a slightly different notation.) So $f \in (A \hookrightarrow B)$ if there is some subset $C \subseteq A$ such that $f \in (C \rightarrowtail B)$. The set $(A \hookrightarrow B)$ can naturally be ordered by inclusion: $f \leq g$ if and only if for all $a \in \text{Domain}(f)$, $a \in \text{Domain}(g)$ and $g(a) = f(a)$. (So considered as sets of ordered pairs, $f \subseteq g$.) It is not hard to see that this ordering is a partial ordering on $(A \hookrightarrow B)$.

(a) Show that if $C \subseteq (A \hookrightarrow B)$ is a chain in this poset, then $\cup C$ is in $(A \hookrightarrow B)$ and is an upper bound of C in the poset.

Solution: We need to show that $\cup C$ is a function with domain some subset of A and image contained in B, and that this function is injective. Since each member of C is a partial injective function from A to B, then the union of C will be a subset of $A \times B$ (since each member of C is).

To show that $\cup C$ is a function, we need to show that if $(a, b) \in \cup C$ for some $a \in A$ and $b \in B$, then if $(a, b') \in C$, we must have that $b = b'$. Since $(a, b) \in cupC$, then $(a, b) \in f$ for some $f \in C$, i.e., $f(a) = b$. Similarly, $(a, b') \in \cup C$ implies that for some $g \in C$, $g(a) = b'$. Since C is a chain, then either $f \subseteq g$ or $g \subseteq f$. In either case, we conclude that $f(a) = g(a)$, which implies that $b = b'$. So $h = \cup C$ is some partial function from A to B. Let C be the domain of h and suppose that for some c, $c' \in C$, $h(c) = h(c') = b$. Then

(c,b), $(c'b) \in \cup C$ and so there is some $k \in C$ with (c,b), $(c',b) \in k$. Since k is injective, we conclude that $c = c'$. So h is an injective partial function from A to B.

Thus $h = \cup C$ is in the poset and is an upper bound for C.

(b) Show that this poset has a maximal element.

Solution: By Zorn's Lemma, this poset has a maximal element, since every chain in the poset has an upper bound.

(c) Let $f \in (A \hookrightarrow B)$ be maximal in this poset. Show that either $\text{Domain}(f) = A$, and so f is an injection from A to B, or $\text{Image}(f) = B$, and so $B =_c \text{Domain}(f)$.

Solution: Suppose that both conditions fail for the maximal element f. Then the domain of f is some proper subset C of A, and the image of f is some proper subset D of B. Let $a \in A \setminus C$ and $b \in B \setminus D$ and define $f' = f \cup \{(a,b)\}$. Then f' is an injective function from $C \cup \{a\}$ onto the set $D \cup \{b\}$, and so is a member of the poset that is strictly larger than f. This contradicts that f is maximal.

So, one of these conditions must hold for f. In the first case, the domain of f is A, which establishes that f is an injection from A to B. In the second case, f is a partial map from A onto B, so, with $C = \text{Domain}(f)$, we have that f is a bijection from C to B, and so $B =_c C = \text{Domain}(f)$.

(d) Conclude that for all sets A and B, either $A \leq_c B$ or $B \leq_c A$.

Solution: From the previous part, using a maximal element f of the poset $(C \rightarrowtail B)$, we see that either f is an injection from A into B, in which case $A \leq_c B$, or that A has a subset, $C = \text{Domain}(f)$, with $C =_c B$. Then $B \leq_c A$.