MATH 4LT3/6LT3 Assignment #5 Solutions
Due: Friday, 24 November, by 11:59pm

Unless otherwise stated, in your solutions you may use the Axiom of Choice, or any equivalent statement that has been discussed in the lectures

1. (a) Show that the Pairset Axiom can be deduced from the Replacement Axiom. Hint: first show that there is some two element set S, using some of the other axioms, and then show, if A and B are sets, that there is some definite unary function h such that $h[S] = \{A, B\}$.

   Solution: We can start with the empty set and apply the power set operation to it, twice, to produce a 2-element set $T = \{\emptyset, \{\emptyset\}\} = \mathcal{P}\mathcal{P}(\emptyset)$. Let $h(x)$ be the following unary definite operation:

   $$h(x) = \begin{cases} A & \text{if } x = \{\emptyset\} \\ B & \text{otherwise} \end{cases}.$$

   Since T is a set and h is a unary definite operation, then

   $$h[T] = \{h(\emptyset), h(\{\emptyset\})\} = \{A, B\}$$

   is also a set.

   (b) Show that the Separation Axiom can be deduced from the Replacement Axiom.

   Solution: Let A be a set and $P(x)$ a unary definite condition. We need to show that $B = \{a \in A \mid P(a)\}$ is also a set. There are two cases to consider. The easy one is when B is empty. Then, trivially, B is a set. If B is non-empty, then let a_0 be some element of B. For this element a_0, Let $h(x)$ be the following definite unary operation:

   $$h(x) = \begin{cases} a & \text{if } P(a) \text{ holds} \\ a_0 & \text{otherwise} \end{cases}.$$

   By the Axiom of Replacement, $h[A]$ is a set. But

   $$h[A] = \{h(a) \mid a \in A\} = \{a \in A \mid P(a)\} = B.$$

2. Let A be a set and $\chi(A) = (h(A), \leq_{\chi(A)})$ be the well order given by Hartog's Theorem. Show that $\leq_{\chi(A)}$ is a best well ordering of the set $h(A)$.

   Solution: We need to show that if $\alpha \in h(A)$ then $seg_{\chi(A)}(\alpha) <_c h(A)$. Let $\alpha \in h(A)$. Then $\alpha = [U/\sim_A]$ for some well ordering U of a subset B of A. By the Lemma found in the proof of Hartog's Theorem (Theorem 7.34), we get that $seg_{\chi(A)}(\alpha) =_o U$ and so $seg_{\chi(A)}(\alpha) =_c B \leq_c A$. By Hartog's Theorem (and the Axiom of Choice) we also have that $A <_c h(A)$ and so $seg_{\chi(A)}(\alpha) <_c h(A)$, as required.

3. Let $\mathcal{N}$ be the class

$$\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}, \dots\}.$$

   So, $\mathcal{N}$ contains the emptyset and satisfies the property that if $x \in \mathcal{N}$ then $x \cup \{x\}$ is also in $\mathcal{N}$. Use the Replacement Axiom to show that $\mathcal{N}$ is a set.

   This question is related to question #8 from Assignment #1. The usual Axiom of Infinity essentially asserts the existence of the set $\mathcal{N}$. In order to show that $\mathcal{N}$ exists as a set using the axioms from the textbook requires the use of the Replacement Axiom.

   Solution: We will show that there is a function f with domain $\mathbb{N}$ and with range $\mathcal{N}$ by applying the version of the recursion theorem stated in Corollary 11.6 (1). The proof of this version of the recursion theorem uses the Axiom of Replacement. Define $H(x, y)$ to be the following definite binary operation:

$$H(x,y) = \begin{cases} x(y-1) \cup \{x(y-1)\} & \text{if } y \in \mathbb{N} \setminus \{0\} \text{ and } x \text{ is a function} \\ & \text{with domain } \{0, 1, \dots, y-1\} \\ \emptyset & \text{otherwise} \end{cases} .$$

   By the recursion theorem, there is a unique function f with domain $\mathbb{N}$ such that for all $n \in \mathbb{N}$, $f(n) = H(f|_{seq_{\mathbb{N}}(n)}, n)$. We show by induction on n that $f(n)$ is the nth element of $\mathcal{N}$. For $n = 0$, $f(0) = H(\emptyset, 0) = \emptyset$.

Assume that $n > 0$ and that the result holds for all smaller values of n. We have that

$$f(n) = H(f|_{seq_{\mathbb{N}}(n)}, n) = f(n-1) \cup \{f(n-1)\}$$

since $f|_{seq_{\mathbb{N}}(n)}$ is a function with domain $\{0, 1, \ldots, n-1\}$. By induction, $f(n-1)$ is the $(n-1)$st element of $\mathcal{N}$ and by definition, the nth entry is $f(n-1) \cup \{f(n-1)\}$, as required.

Since f is a function with domain $\mathbb{N}$, then $f[\mathbb{N}]$ is also a set. But $f[\mathbb{N}] = \mathcal{N}$.

4. Using the Axiom of Regularity (or the Principle of Foundation, or the Axiom of Foundation) show that sets with the following properties cannot exist:

   (a) A set A such that $A = \{A\}$.

   Solution: Given such a set, define f to be the function with domain $\mathbb{N}$ and range $\{A\}$ such that $f(n) = A$ for all n. Then for all $n > 0$, $f(n) \in f(n-1)$. By definition, A is illfounded. By the Axiom of Foundation, such a set cannot exist.

   (b) For some $n > 0$, a sequence of sets A_i, $0 \leq i \leq n$ such that $A_{i+1} \in A_i$, for $0 \leq i < n$, and $A_1 = A_n$.

   Solution: The following infinite sequence shows that A_0 is ill founded:

$$A_0 \ni A_1 \ni \cdots \ni A_{n-1} \ni A_n = A_1 \ni A_2 \ni \cdots$$

5. Use the Axiom of Regularity to show that the construction from question #1 (b) of Assignment #2 satisfies the ordered pair property (OP1). It also satisfies (OP2), but you don't need to show that.

Solution: To see that (OP1) is satisfied, suppose that $\{x, \{x, y\}\} = \{x', \{x', y'\}\}$. Using the Foundation Axiom, it follows that $x \neq \{x, y\}$ and $x' \neq \{x', y'\}$, since if, for example, $x = \{x, y\}$ then we have that $x \in x$, which implies that x is ill founded. So both $\{x, \{x, y\}\}$ and $\{x', \{x', y'\}\}$ are 2-element sets. By the axiom of extensionality, either

$x = x'$ and $\{x, y\} = \{x', y'\}$ or $x = \{x', y'\}$ and $x' = \{x, y\}$. In the latter case, we have that $x' \in x \in x'$, which implies that x' is ill founded. So the former case must hold. From $x = x'$ and $\{x, y\} = \{x', y'\}$ it follows that $y = y'$.

Bonus Question: For κ a cardinal, the cofinality of κ, denoted $cf(\kappa)$, is given in Definition 9.23 of the textbook.

1. Show that $cf(\aleph_0) = \aleph_0$ and that $cf(\aleph_1) = \aleph_1$.

   Solution: By definition, $\aleph_0 = \omega = Ord(\mathbb{N})$. Since $\mathbb{N}$ is not equal to the union of a finite number of finite sets, then $cf(\aleph_0) \geq_c \aleph_0$. Also since $\mathbb{N}$ can be expressed as a countable union of finite sets, then $cf(\aleph_0) \leq_c \aleph_0$ (in general, $cf(\kappa) \leq_c \kappa$) and so $cf(\aleph_0) = \aleph_0$.

   To show $cf(\aleph_1) = \aleph_1$ we just need to show that $cf(\aleph_1) \geq_c \aleph_1$ or that $cf(\aleph_1) >_c \aleph_0$. This follows, since any countable union of countable sets is countable, so $\aleph_1$, being uncountable, can't be expressed in this manner.

2. The gimel function on the class of cardinals is defined by: $\gimel(\kappa) = \kappa^{cf(\kappa)}$. Use König's Theorem to show that for any cardinal κ, $\kappa <_c \gimel(\kappa)$.

   Solution: Let $\gamma = cf(\kappa)$. Then there are disjoint sets K_i, $i \in \gamma$ such that $K_i <_c \kappa$ and $\kappa =_c \bigcup_{i\in\gamma} K_i$. Then by König's Theorem,

$$\kappa =_c \bigcup_{i\in\gamma} K_i <_c \prod_{i\in\gamma} \kappa = \kappa^{\gamma} = \kappa^{cf(\kappa)}.$$