MATH 4LT3/6LT3 Assignment #6

Note: This assignment will not be graded. You are strongly encouraged to work on the following problems and to compare your solutions to the posted solutions. At least one of the following questions will appear on the upcoming final exam. Unless otherwise stated, in your solutions you may use any of the ZFC axioms.

1. (a) Let $\mathcal{E}$ be a set of transitive sets. Show that $\bigcap \mathcal{E}$ is a transitive set.

   Solution: A set A is transitive if whenever $x \in A$, then $x \subseteq A$, or equivalently, whenever $x \in A$ and $y \in x$, then $y \in A$.

   Let $x \in \bigcap \mathcal{E}$ and $y \in x$. We need to show that $y \in \bigcap \mathcal{E}$, i.e., that $y \in A$ for all $A \in \mathcal{E}$. Let $A \in \mathcal{E}$. Since $x \in \bigcap \mathcal{E}$ then $x \in A$ and since A is transitive, then $y \in A$, as required.

   (b) Show that every von Neumann ordinal is a transitive set.

   Solution: Let α be an ordinal. Then by definition, $\alpha = v_U[A]$ for some well ordered set $U = (A, \leq)$, where v_U is the von Neumann map for U.

   Then $\alpha = \{v_U(u) \mid u \in A\}$ and for all $u \in A$, $v_U(u) = \{v_U(v) \mid v < u\}$. So if $x \in \alpha$, then $x = v_U(u)$ for some $u \in U$. Let $y \in x$. Then $y \in v_U(u)$, so $y = v_U(v)$ for some $v < u$ in A. But then $y \in \alpha$, showing that $x \subseteq \alpha$. Thus α is transitive.

   (c) For A a set, define the following sequence of sets: $T_0(A) = \{A\}$, and given $T_n(A)$, $T_{n+1}(A) = \bigcup T_n(A)$. Let $T(A) = \bigcup_{n \geq 0} T_n(A)$.

   i. Explain why the function that send $n \in \mathbb{N}$ to $T_n(A)$ exists.

   Solution: The existence of this sequence follows from the Recursion Theorem (Corollary 11.6 (1)). The definite unary operation $H(x, y)$ to use is

$$H(x,y) = \begin{cases} \{A\} & \text{if } y = 0 \\ \bigcup x(y-1) & \text{if } y \in \mathbb{N} \setminus \{0\} \text{ and} \\ & x \text{ is a function with } y-1 \text{ in its domain} \\ \emptyset & \text{otherwise} \end{cases}.$$

ii. Prove that $T(A)$ is a transitive set that contains A.

Solution: Since in general, for any set X, if $y \in X$, then $y \subseteq \bigcup X$, then for all n, if $x \in T_n(A)$, then $x \subseteq \bigcup T_n(A) = T_{n+1}(A)$. So if $x \in T(A)$, then $x \in T_n(A)$ for some $n \geq 0$, and then $x \subseteq T_{n+1}(A) \subseteq T(A)$.

iii. Show that if M is any transitive set that contains A then $T(A) \subseteq M$.

Solution: It suffices to show that for all n, $T_n(A) \subseteq M$. By assumption, this holds for $n = 0$. Assume that it holds for $T_n(A)$. Since M is transitive, then for each $x \in T_n(A) \subseteq M$, $x \subseteq M$. If $y \in T_{n+1}(A)$, then $y \in x$ for some $x \in T_n(A)$. Since $x \subseteq M$, then $y \in M$ and so $T_{n+1}(A) \subseteq M$.

Note that this shows that $T(A)$ is the smallest transitive set that contains A and so is called the transitive closure of A. Another way to show that such a set exists, is to observe that since A is a set, then for some ordinal α, $A \in \mathcal{V}_\alpha$. Since we've shown that this set is transitive, and from part (a) that the intersection of a set of transitive sets is transitive, then the smallest transitive set that contains A is equal to the intersection of all transitive subsets of $\mathcal{V}_\alpha$ that contain A.

Here is another characterization of $T(A)$. We claim that for a given n, $x \in T_n(A)$ if and only if there are sets $A = x_0 \ni x_1 \ni \cdots \ni x_{n-1} \ni x_n = x$. Call this an $\in$-chain of length n. We can prove this by induction. For $n = 0$, the claim holds, since $T_0(A) = \{A\}$. Suppose that it holds for n and consider $T_{n+1}(A)$. We have that for a set x,

- $x \in T_{n+1}(A)$ if and only if
- $x \in y$ for some $y \in T_n(A)$ if and only if
- for some sets x_i, $A = x_0 \ni x_1 \ni \cdots \ni x_{n-1} \ni x_n = y$ and $x \in y$.

So $x \in T_{n+1}(A)$ if and only if an $\in$-chain of length $n+1$ exists that starts with A and ends with x. So, $T(A)$ is the set of all sets x that can be obtained in this way, for some n.

2. (a) Let $U = (A, \leq)$ be a best wellordered set. Show that $|A| = ord(U)$. Here, $|A|$ denotes the von Neumann cardinal of the set A.

Solution: $U = (A, \leq)$ is a best wellordered set if it is a well ordered set such that for all $a \in A$, $|\text{seg}_U(a)| <_c |A|$. Here $\text{seg}_U(a)$ is the proper initial segment of U consisting of all elements $< a$.

We know that $|A|$ is the least ordinal α such that $A =_c \alpha$. Since U is a well order on A then $ord(U) =_c A$ and so $|A| \leq ord(U) =_o U$. If $|A| < ord(U) =_o U$ then the ordinal $|A|$ is order isomorphic to a proper initial segment of the ordinal $ord(U)$ and hence of U (since this is one way to describe the ordering on the class of ordinals). Since $\leq$ is a best wellordering of A, then every proper initial segment of U is $<_c A$ and so we conclude that $|A| <_c A$, a contradiction. So $|A| = ord(U)$.

(b) Find a well ordered set $V = (B, \leq)$ such that $|B| \neq ord(V)$.

Solution: We just need to find a well ordering of a set B that isn't a best wellorder. For example, let B be the ordinal $\omega + 1$. Then $ord(B) = B = \omega + 1$ and since B is a countably infinite set, $|B| = \omega \neq ord(B)$.

(c) Show that the class of all von Neumann ordinals, ON, is not a set.

Solution: Suppose that ON is a set. Then ON is an ordinal, since it is transitive and $\in$-connected. To see this, suppose that $\alpha \in \text{ON}$ and $\beta \in \alpha$. Then β is an ordinal and so $\beta \in \text{ON}$. So ON is a transitive set. ON is $\in$-connected, since if α, $\beta \in \text{ON}$ then either $\alpha = \beta$, $\alpha \in \beta$ or $\beta \in \alpha$. Since ON is an ordinal and ON contains all ordinals, then $\text{ON} \in \text{ON}$, which contradicts the Axiom of Foundation.

Alternatively, if ON is a set, then we've shown that it is an ordinal, and then so is $ON + 1$. But then $ON + 1 \in ON$ and then we get that some proper initial segment of ON is order isomorphic to ON. This can't happen in a well ordered set.

(d) Show that the class of all von Neumann cardinals, Card_v, is not a set.

Solution: Suppose that Card_v is a set. Then so is the set $A = \bigcup \text{Card}_v$. (Note that A is actually an ordinal, since the supremum of any set of ordinals, is the union of the set, and is an ordinal, since it is transitive and $\in$-connected.) If κ is a cardinal, then

$\kappa \in \text{Card}_v$ and so $\kappa \subseteq A$. So, for all cardinals κ, $\kappa \leq_c A$. In particular, $\mathcal{P}(A) =_c |\mathcal{P}(A)| \leq_c A$. But we know from Cantor that $A <_c \mathcal{P}(A)$ and so we get the contradiction that $\mathcal{P}(A) <_c \mathcal{P}(A)$. Thus A, and hence Card_v are not sets.

Another way to see this is to use the Axiom of Replacement and part (c). Consider the definite unary operation that maps a cardinal κ to the unique ordinal α such that $\kappa = \aleph_\alpha$. We saw in class that every cardinal is an aleph, so this operation is well defined; call it $H(x)$. If Card_v is a set, then $H[\text{Card}_v]$ is also a set, but this is the class of all ordinals, which is not a set.

3. Recall the definition of $\text{rank}(A)$, the rank of the set A.

   (a) What are the ranks of $\mathbb{N}$ and $\mathcal{P}(\mathbb{N})$?

   Solution: The rank of a set A is the least ordinal α such that $A \in \mathcal{V}_{\alpha+1}$ and so is the least α such that $A \subseteq \mathcal{V}_\alpha$.

   This can be shown more directly by using part (b) of this question, since the rank of the nth member of $\mathbb{N}$ is n (shown below, by induction on n) and so the rank of $\mathbb{N}$ is the supremum of the set of natural numbers, and so is ω.

   If we use that $\mathbb{N} = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \ldots\}$ (the version of the natural numbers given by Zermelo's infinity axiom), then we will show that $\text{rank}(\mathbb{N}) = \omega$. We first note that this rank cannot be equal to n for some $n \in \mathbb{N}$, since then $\mathbb{N} \subseteq \mathcal{V}_n$. But this is a finite set, which would imply that $\mathbb{N}$ is also finite. So, $\text{rank}(\mathbb{N}) \geq \omega$.

   To see that it is equal to ω, it suffices to show that $\text{rank}(\mathbb{N}) \leq \omega$ or equivalently that $\mathbb{N} \subseteq \mathcal{V}_\omega$. We can show this by proving that for $n \in \mathbb{N}$, $\text{rank}(n) = n$, or equivalently that $n \subseteq \mathcal{V}_n$. This can be shown by induction on n: for $n = 0$, $n = \emptyset$ and then $n \subseteq \mathcal{V}_0 = \emptyset$. Suppose that $n \subseteq \mathcal{V}_n$, or equivalently that $n \in \mathcal{V}_{n+1}$. Then $n + 1 = \{n\} \subseteq \mathcal{V}_{n+1}$, as required.

   So for all $n \in \mathbb{N}$, $\text{rank}(n) = n$, and so $n \in \mathcal{V}_\omega$. But then $\mathbb{N} \subseteq \mathcal{V}_\omega$, showing that $\text{rank}(\mathbb{N}) \leq \omega$.

   Since $\mathbb{N} \subseteq \mathcal{V}_\omega$, then $\mathcal{P}(\mathbb{N}) \subseteq \mathcal{V}_{\omega+1}$, since each member of $\mathcal{P}(\mathbb{N})$ is a subset of $\mathcal{V}_\omega$, and so is a member of $\mathcal{V}_{\omega+1}$. So $\text{rank}(\mathcal{P}(\mathbb{N})) \leq \omega+1$. This rank can't be less than $\omega + 1$, since this would imply that

$\mathcal{P}(\mathbb{N}) \subseteq \mathcal{V}_\omega$, which then would imply that $\mathbb{N}$, a member of $\mathcal{P}(\mathbb{N})$, would have finite rank. So $\text{rank}(\mathcal{P}(\mathbb{N})) = \omega + 1$.

(b) Prove that

- if $x \in A$, then $\text{rank}(x) < \text{rank}(A)$.
- $\text{rank}(A) = \sup\{\text{rank}(x) + 1 \mid x \in A\}$.

Solution: Let $\alpha = \text{rank}(A)$. A is non-empty and so $\alpha > 0$. We have that $A \subseteq \mathcal{V}_\alpha$ and so $x \in \mathcal{V}_\alpha$, which implies that $x \subseteq \mathcal{V}_\beta$ for some $\beta < \alpha$. Thus $\text{rank}(x) \leq \beta < \alpha = \text{rank}(A)$.

Let $\beta = \sup\{\text{rank}(x) + 1 \mid x \in A\}$ and $\alpha = \text{rank}(A)$. For $x \in A$, $x \in \mathcal{V}_{\text{rank}(x)+1}$ (this holds for any set x, by the definition of rank) and so $x \in \mathcal{V}_\beta$. Thus $A \subseteq \mathcal{V}_\beta$ and so $\alpha = \text{rank}(A) \leq \beta$.

If $x \in A$, then by the previous part, $\text{rank}(x) + 1 \leq \text{rank}(A) = \alpha$ and so $\beta \leq \alpha$. Thus $\alpha = \beta$.

(c) Show that for α an ordinal, $\text{rank}(\alpha) = \alpha$.

Solution: We prove this by induction on α. For $\alpha = 0$, this is immediate. Suppose it holds for α, and consider $\alpha + 1$. We have that $\alpha + 1 = \alpha \cup \{\alpha\}$ and so by part (b),

$$\text{rank}(\alpha + 1) = \sup\{\text{rank}(x) + 1 \mid x \in \alpha \cup \{\alpha\}\} = \alpha + 1,$$

since $\text{rank}(\alpha) + 1 = \alpha + 1$ and for all $x \in \alpha$, $\text{rank}(x) + 1 \leq \alpha$.

If α is a limit ordinal, then $\alpha = \{\beta \in ON \mid \beta \in \alpha\}$ and then by induction,

$$\text{rank}(\alpha) = \sup\{\text{rank}(\beta) + 1 \mid \beta \in \alpha\} = \sup\{\beta + 1 \mid \beta \in \alpha\} = \alpha.$$

4. Recall the cumulative hierarchy, $\mathcal{V} = \bigcup_{\alpha \in ON} \mathcal{V}_\alpha$. We saw that all sets belong to $\mathcal{V}$.

(a) Show that if $A \in \mathcal{V}_\omega$, then A and $T(A)$ are finite sets and $T(A) \in \mathcal{V}_\omega$. (See question #1.)

Solution: Since $A \in \mathcal{V}_\omega$ then $A \in \mathcal{V}_n$ for some $n > 0$ and so $A \subseteq \mathcal{V}_{n-1}$. By induction, it is easy to see that each $\mathcal{V}_k$ is a finite set, for $k \in \mathbb{N}$, and so A is finite, since it is a subset of a finite set. Furthermore, we saw in class that each $\mathcal{V}_n$ is transitive and so by Question #1, $T(A) \subseteq \mathcal{V}_n$ and hence that $T(A) \in \mathcal{V}_{n+1}$, which implies that $T(A) \in \mathcal{V}_\omega$. It also follows that $T(A)$ is finite, since it is a subset of $\mathcal{V}_n$, a finite set.

(b) Show that for A a set, if $T(A)$ is a finite set, then $A \in \mathcal{V}_\omega$.

Solution: We will show, by induction on n, that if $\text{rank}(A) \geq n$, then $|T(A)| \geq n$. From this the result follows, since by the contrapositive, if $|T(A)| < n$ for some natural number n, then $\text{rank}(A) < n$, and from this it follows that $A \in \mathcal{V}_\omega$.

We prove that if $\text{rank}(A) \geq n$, then $|T(A)| \geq n$, by induction on n. The claim holds for $n = 0$. Suppose that it holds for n and let A be a set with rank $\geq n + 1$. Then by part (b) of the previous question, there is some $x \in A$ that has rank $\geq n$ (or else all member of A have rank less than n, which by part (b) implies that the rank of A is at most n). But then $|T(x)| \geq n$ and since $T(x) \subseteq T(A)$, we have that $|T(A)| \geq n$ as well. But the set A belongs to $T(A)$ and not to $T(x)$ and so $|T(A)| \geq n + 1$, as required.

To see that $A \notin T(x)$, we can use the alternate characterization of $T(x)$ from earlier. If A is in $T(x)$, then there is an $\in$-chain starting at x and ending at A. But since $x \in A$, we contradict the Axiom of Foundation.

(c) Find a finite set B such that $B \notin \mathcal{V}_\omega$.

Solution: From the previous part, we just need to find a finite set B such that $T(B)$ is infinite. For any infinite set A, let $B = \{A\}$. Then B is finite (it only has one element), but the set A is a subset of $T(B)$, and so $B \notin \mathcal{V}_\omega$.

(d) Show that the set $\mathcal{V}_\omega$ satisfies all of the ZFC axioms, except for the Axiom of Infinity and the Replacement Axiom.

Solution:

(I) **Axiom of Extensionality.** This hold for all sets, and so in particular it holds for sets in $\mathcal{V}_\omega$.

(II) **Emptyset and Pairset Axiom.** $\emptyset$ is in $\mathcal{V}_\omega$, and if A, $B \in \mathcal{V}_\omega$, say they belong to $\mathcal{V}_n$, then $\{A, B\}$ belongs to $\mathcal{V}_{n+1}$.

(III) **Separation Axiom.** If A belongs to $\mathcal{V}_n$ for some n and B is a subset of A defined using this axiom, then B will also belong to $\mathcal{V}_n$.

(IV) **Powerset Axiom.** If A is in $\mathcal{V}_n$ then $\mathcal{P}(A)$ belongs to $\mathcal{V}_{n+1}$.

(V) **Unionset Axiom.** If $A \in \mathcal{V}_n$, then $A \subseteq \mathcal{V}_n$ since this set is transitive, and so for each $x \in A$, $x \in \mathcal{V}_n$. Thus if $x \in A$, then $x \subseteq \mathcal{V}_n$, since this set is transitive. But then the union of these sets, namely $\bigcup A$, is also a subset of $\mathcal{V}_n$, and so $\bigcup A \in \mathcal{V}_{n+1}$.

(VII) **Axiom of Choice.** If f is a choice function for some $P \subseteq A \times B$ with A and B in $\mathcal{V}_\omega$, then $A \times B$ is also in $\mathcal{V}_\omega$ and so f, being a subset of this product, is also in $\mathcal{V}_\omega$.

(IX) **Axiom of Foundation.** Every set is grounded, and so every set in $\mathcal{V}_\omega$ is as well.

Note that the members of $\mathcal{V}_\omega$ are called the hereditarily finite sets.

5. Consider the following sequence, indexed by the ordinals, of cardinals:

$$\begin{aligned} \beth_0 &= \aleph_0 = |\mathbb{N}|, \\ \beth_{\beta+1} &= 2^{\beth_\beta}, \\ \beth_\lambda &= \sup\{\beth_\beta \mid \beta < \lambda\}, \text{ if } \lambda \text{ is a limit ordinal.} \end{aligned}$$

(a) Justify the existence of this sequence.

Solution: This sequence is indexed by the ordinals and has a recursive definition, and so it follows from the Recursion theorem (Theorem 12.18) that this sequence of sets exists.

(b) Show that $|\mathcal{V}_\omega| = \beth_0$.

Solution: $\mathcal{V}_\omega$ is a countable union of countable sets and so is countable. Thus $|\mathcal{V}_\omega| = \aleph_0 = \beth_0$.

(c) Show that for any ordinal α, $|\mathcal{V}_{\omega+\alpha}| = \beth_\alpha$. For the definition of ordinal addition used here, consult Theorem 12.19.

Solution: We can show this by induction on α. The previous part establishes the base. Suppose that it holds for α. Then

$$\mathcal{V}_{\omega+(\alpha+1)} = \mathcal{V}_{(\omega+\alpha)+1} = \mathcal{P}(\mathcal{V}_{\omega+\alpha}),$$

and so

$$|\mathcal{V}_{\omega+(\alpha)+1)}| = |\mathcal{P}(\mathcal{V}_{\omega+\alpha})| = 2^{|\mathcal{V}_{\omega+\alpha}|} = 2^{\beth_\alpha} = \beth_{\alpha+1}.$$

For λ a limit ordinal, $\mathcal{V}_{\omega+\lambda} = \bigcup_{\beta<\lambda} \mathcal{V}_{\omega+\beta}$ and so

$$|\mathcal{V}_{\omega+\lambda}| = |\bigcup_{\beta<\lambda} \mathcal{V}_{\omega+\beta}| = \sup\{|\mathcal{V}_{\omega+\beta}| \mid \beta < \lambda\} = \sup\{\beth_\beta \mid \beta < \lambda\} = \beth_\lambda.$$