A Residually small, finitely generated, semi-simple variety which is not residually finite *

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1 Introduction

In [1] it is asked whether or not there exists a finite algebra $\mathbf{A}$ such that $\mathsf{V}(\mathbf{A})$ admits no finite bound on the size of its simple members but which nevertheless admits some cardinal bound on the size of these algebras. In this article we present an example of such an algebra. To be precise, we will describe an 8 element algebra $\mathbf{A}'$ which generates a semi-simple variety which contains a countably infinite simple algebra into which all other infinite simple algebras in the variety can be embedded.

The algebra $\mathbf{A}'$ which we will describe is derived from an algebra discovered by R. McKenzie and presented in [2]. McKenzie's algebra (which we will call $\mathbf{A}$) generates a residually small variety which has a countably infinite subdirectly irreducible algebra which, up to isomorphism, is the only infinite subdirectly irreducible algebra in the variety. Thus our example is only a modest improvement of his.

Our algebra $\mathbf{A}'$ is obtained from the algebra $\mathbf{A}$ of section 6 in [2] by adjoining the following three new basic operations: $x \circ y$, $x \leftarrow y$ and $T_\circ(x, y, z, u)$.

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Define $x \circ y$ on A so that

$$1 \circ C = H \circ D = C, \quad 2 \circ D = D$$
$$1 \circ \bar{C} = H \circ \bar{D} = \bar{C}, \quad 2 \circ \bar{D} = \bar{D}$$

and so that $x \circ y = 0$ for all other values of x and y.

Define $x \leftarrow y$ on A so that

$$1 \leftarrow C = 1 \leftarrow \bar{C} = 1$$
$$H \leftarrow C = H \leftarrow \bar{C} = H$$
$$2 \leftarrow D = 2 \leftarrow \bar{D} = 2$$

and so that $x \leftarrow y = 0$ for all other values of x and y.

Finally, define $T_\circ$ by

$$T_\circ(x,y,z,u) = \begin{cases} 0 & \text{unless } x \circ y = z \circ u \neq 0 \\ x \circ y & \text{if } x \circ y \neq 0,\ x = z \text{ and } y = u \\ \overline{x \circ y} & \text{if } x \circ y = z \circ u \neq 0,\ x \neq z \text{ or } y \neq u \end{cases}$$

Note that all of the above operations are monotone with respect to the semilattice order on $\mathbf{A}'$.

Following [2] we will call a subdirectly irreducible member of $\mathsf{V}(\mathbf{A}')$ **large** if it is not in $\mathsf{HS}(\mathbf{A}')$. The following Theorem summarizes the properties of $\mathbf{A}'$ which we will explore in this article.

THEOREM 1.1 *The variety $\mathsf{V}(\mathbf{A}')$ is a semi-simple variety which contains a countably infinite simple algebra $\mathbf{Q}'_Z$ such that every large subdirectly irreducible algebra in $\mathsf{V}(\mathbf{A}')$ can be embedded into $\mathbf{Q}'_Z$.*

The algebra $\mathbf{Q}'_Z$ referred to in the above Theorem is obtained by adding the following three basic operations to the algebra $\mathbf{Q}_Z$ defined in Definition 6.1 of [2]:

$$x \circ y = \begin{cases} b_{n+1} & \text{if } x = a_n \text{ and } y = b_n \\ 0 & \text{otherwise} \end{cases}$$

$$x \leftarrow y = \begin{cases} a_n & \text{if } x = a_n \text{ and } y = b_{n+1} \\ 0 & \text{otherwise} \end{cases}$$

and

$$T_\circ(x,y,z,u) = (x \circ y) \wedge (z \circ u).$$

It is straightforward to verify that $\mathbf{Q}'_Z$ is a simple algebra and, by following the proof of Lemma 6.1 of [2], is a member of $\mathsf{V}(\mathbf{A}')$.

2 Finite Subdirectly Irreducible Algebras

In this section we will analyze the structure of the finite subdirectly irreducible algebras in $\mathsf{V}(\mathbf{A}')$. We will show that every such algebra is either a member of $\mathsf{HS}(\mathbf{A}')$ or is isomorphic to $\mathbf{Q}'_n$ for some $n \geq 1$, where $\mathbf{Q}'_n$ is the subalgebra of $\mathbf{Q}'_Z$ with universe

$$\{0, a_0, \ldots, a_n, b_0, \ldots, b_n, b_{n+1}\}.$$

We leave it to the reader to verify that the $\mathbf{Q}'_n$ are simple algebras and that every subdirectly irreducible member of $\mathsf{HS}(\mathbf{A}')$ is simple.

For the remainder of this section, let $\mathbf{S}$ be a finite large subdirectly irreducible algebra in $\mathsf{V}(\mathbf{A}')$. So there is some finite set T of minimal size, some algebra $\mathbf{B} \subseteq (\mathbf{A}')^T$ and some $\theta \in \operatorname{Con} \mathbf{B}$ such that $\mathbf{S} \simeq \mathbf{B}/\theta$. Note that since $\mathbf{S}$ is assumed to be large then T has at least two elements. Let $\bar{\theta}$ be the unique cover of θ in $\operatorname{Con} \mathbf{B}$.

Since Lemma 6.4 of [2] is proved using the height one semilattice structure on $\mathbf{B}$ and the minimality of the size of T it also holds for our algebra $\mathbf{B}$. As in [2] we define the set B_1 as follows:

$$\begin{aligned} B_1 = \{u \in B \ :\ & u = p \text{ or for some } n \geq 1 \text{ there exists } x_0, \ldots, x_n \in B \\ & \text{with } u = x_i \text{ for some } i \leq n \text{ and } x_0 \cdot x_1 \cdots x_n = p\} \end{aligned}$$

Lemmas 6.6 and 6.7 of [2] and their proofs are valid for our algebra $\mathbf{B}$ and in addition the following variant of Lemma 6.7 is also true:

$$\text{if } a \circ b = c \circ d \in B_1 \text{ then } a = c \in U^T \setminus \{1, 2\}^T \text{ and } b = d \in W^T.$$

This can be proved by using the operation $T_\circ$ in place of the operation T in the proof of Lemma 6.7 found in [2].

We claim that the element p from Lemma 6.4 and used in the definition of B_1 can be chosen so that it cannot be written in the form $u \leftarrow v$ or $u \circ v$ for any elements u, v from B.

To prove this we will use some of the following easy to verify properties of the basic operations of $\mathbf{A}'$. For u, v and $w \in A$,

1. $u \cdot v \neq 0$ if and only if $u \leftarrow v \neq 0$ and if $u \cdot v \neq 0$ then $u \leftarrow v = u$ and $u \circ (u \cdot v) = v$,

2. if $u \circ v \neq 0$ then $u \cdot (u \circ v) = v$,

3. $(u \leftarrow v) \cdot v = u \cdot v$ and if $(u \leftarrow v) \cdot w \neq 0$ then $v \in \{w, \bar{w}\}$,

4. if $(u \leftarrow v) \circ w \neq 0$ then $u \circ w \in \{v, \bar{v}\}$,

5. $u \leftarrow (v \leftarrow w) = 0$,

6. $(u \circ v) \circ w = 0 = (u \circ v) \cdot w$.

Using the last item on our list, it is clear that the only iterated products formed with the operation $\circ$ are the fully right-associated ones. Following the practice established in [2] for terms involving $\cdot$ we will denote the (right-associated) product $x_0 \circ (x_1 \circ (\cdots (x_{k-2} \circ (x_{k-1} \circ x_k)) \cdots))$ by $x_0 \circ x_1 \circ \cdots \circ x_k$.

If the element p obtained from Lemma 6.4 happens to be of the form $a \leftarrow b$ then we can replace the element p by b and the element q by $c = q \circ (q \cdot b)$. Since $p(s) \neq 0$ for all $s \in T$ it follows that $p = a$ and that $b(s) \neq 0$ for all $s \in T$ as well. Also, by using some of the above properties of the basic operations, we see that $p \circ (p \cdot b) = b$ and so the pair $\{p, q\}$ is polynomially isomorphic to the pair $\{b, c\}$ via the polynomials $f(x) = x \circ (x \cdot b)$ and $g(x) = p \leftarrow x$ of $\mathbf{B}$. Thus b and c satisfy all of the properties of p and q listed in Lemma 6.4 and (by item 5 above) b is not of the form $u \leftarrow v$.

Now assume that p can be written as $u \circ v$ for some elements u, v from B. The proof of Lemma 6.9 using Lemma 6.7 for the operation $\circ$ yields that if $p = u_0 \circ u_1 \circ \cdots \circ u_m$ for some u_i in B then all of the u_i's must be distinct. So we can choose m maximal so that $p = u_0 \circ u_1 \circ \cdots \circ u_m$ for some $u_i \in B$. We claim that we can replace the elements p and q by the elements u_m and v where $v = u_{m-1} \cdot u_{m-2} \cdots u_0 \cdot q$.

Using the properties of $x \cdot y$ and $x \circ y$ noted earlier we see that the polynomial $f(x) = u_{m-1} \cdot u_{m-2} \cdots u_0 \cdot x$ maps p to u_m and q to v. On the other hand, the polynomial $g(x) = u_0 \circ u_1 \circ \cdots \circ x$ maps u_m to p and v to q. Thus u_m and v satisfy all of the properties of p and q listed in Lemma 6.4 and, by the maximality of m, u_m cannot be written in the form $a \circ b$ for any elements a, b from B. Also note that since u_m is in the range of the operation $\cdot$ and 0 is the only element in the intersection of the ranges of $\cdot$ and $\leftarrow$ then the element u_m cannot be written as $u \leftarrow v$ either.

Lemma 6.8 of [2] also holds for our algebra $\mathbf{B}$ and it can be proved by considering three new cases in addition to the ones found in the original proof.

Recall that in that proof it is assumed that we have a nonconstant polynomial $f(x)$ of minimal degree such that there is some element $u \in B \setminus B_1$ with $f(u) \in B_1$. The three new cases correspond to considering the possibilities that $f(x)$ is of the form $g(x) \circ h(x)$, $g(x) \leftarrow h(x)$ or $T_\circ(g(x), h(x), k(x), l(x))$ for some polynomials $g(x)$, $h(x)$, $k(x)$ and $l(x)$.

If $f(x) = g(x) \circ h(x)$ then since $f(u) \in B_1$ we have that either $g(u) \circ h(u) = p$ or for some elements $a_0, a_1, \ldots, a_{n-1}$ from B, and $i < n$

$$a_0 \cdot a_1 \cdots a_i \cdot (g(u) \circ h(u)) \cdot a_{i+1} \cdots a_{n-1} = p.$$

The first of these two possibilities can be ruled out since we have selected the element p so that it can't be written in the form $a \circ b$ for any elements a, b from B. In the second case, note that by the last item in our list of properties of the basic operations of $\mathbf{A}'$ it follows that $i = n - 1$ and so $p = a_0 \cdot a_1 \cdots a_{n-1} \cdot (g(u) \circ h(u))$. We deduce that

$$a_{n-1} \circ a_{n-2} \circ \cdots \circ a_0 \circ p = g(u) \circ h(u)$$

to conclude that $a_{n-1} = g(u)$. From this it follows that $a_{n-1} \cdot (g(u) \circ h(u)) = h(u)$ and so both $g(u)$ and $h(u)$ have been shown to belong to B_1. Since at least one of $g(x)$ and $h(x)$ must be nonconstant, we have contradicted the minimality of the degree of $f(x)$.

In the case that $f(x) = g(x) \leftarrow h(x)$ we again have two cases to consider. The first being that $f(u) = p$. This case is not possible since we have assumed that the element p is not of the form $a \leftarrow b$ for any elements a, b from B.

The remaining case is when for some $n \geq 1$ and some $a_i \in B$, $i \leq n$ we have $a_0 \cdot a_1 \cdots a_n = p$ and $f(u) = a_i$ for some $i \leq n$. Since the range of $\leftarrow$ is the set U it follows that $i < n$ and that for some element $c \in B_1$ we have that $(g(u) \leftarrow h(u)) \cdot c$ is in B_1.

Since the element $(g(u) \leftarrow h(u)) \cdot c$ does not have the element 0 in its range then by one of the properties of $\leftarrow$ noted earlier we conclude that for all $s \in T$, $h(u)(s) \in \{c(s), \overline{c(s)}\}$. From Lemma 6.6 we then have that $h(u) = c$ since $c \in B_1$. Thus $h(u) \in B_1$ and, by item 3 of our list of properties of the basic operations of $\mathbf{A}'$,

$$(g(u) \leftarrow h(u)) \cdot c = g(u) \cdot h(u)$$

showing that $g(u) \in B_1$ too. This again leads to a contradiction.

One can handle the case $f(x) = T_\circ(g(x), h(x), k(x), l(x))$ in the manner that the case $f(x) = T(g(x), h(x), k(x), l(x))$ from the proof of Lemma 6.8 is handled.

We are now in a position to show that the algebra $\mathbf{S}$ is isomorphic to $\mathbf{Q}'_n$ for some n, and hence that it can be embedded into $\mathbf{Q}'_Z$. Following Lemma 6.9, we see that $u/\theta = \{u\}$ for all $u \in B_1$ and that $B \setminus B_1 = 0_\mathbf{B}/\theta$. Also, as in the proof of Lemma 6.9 we get that

$$B_1 = \{r_0, \ldots, r_{k-1}, s_0, \ldots, s_k\}$$

with $s_0 = p$ and $r_i \cdot s_{i+1} = s_i$ for $i < k$. We can use these equalities to discover the behaviour of the operations $\circ$ and $\leftarrow$ on B_1. For $i < k$,

$$r_i \circ s_i = r_i \circ (r_i \cdot s_{i+1}) = s_{i+1}$$

and since $r_i \cdot s_{i+1} = s_i$ then

$$r_i \leftarrow s_{i+1} = r_i.$$

It is also the case that any other combination of elements from B_1 using $\circ$ or $\leftarrow$ lies in $B \setminus B_1$.

We leave it to the reader to verify that in $\mathbf{B}$, $T_\circ(x, y, z, u)$ is θ-related to $(x \circ y) \wedge (z \circ u)$ for all x, y, z, $u \in B$. If $k < 2$ then it can be shown that $\mathbf{S}$ lies in $\mathsf{HS}(\mathbf{A}')$, contradicting the fact that T has at least 2 elements, so $k \geq 2$ and thus $\mathbf{S}$ is isomorphic to the algebra $\mathbf{Q}'_k$.

3 Infinite Subdirectly Irreducibles

We conclude our presentation by showing that every infinite subdirectly irreducible algebra in $\mathsf{V}(\mathbf{A}')$ is simple and can be embedded into $\mathbf{Q}'_Z$. In fact we will show that such an algebra must be isomorphic to one of the algebras $\mathbf{Q}'_\omega$, $\mathbf{Q}'_{-\omega}$ or $\mathbf{Q}'_Z$, where $\mathbf{Q}'_\omega$ is the subalgebra of $\mathbf{Q}'_Z$ with universe $\{0, a_0, a_1, \ldots, b_0, b_1, \ldots\}$ and $\mathbf{Q}'_{-\omega}$ is the subalgebra of $\mathbf{Q}'_Z$ with universe $\{0, a_{-1}, a_{-2}, \ldots, b_0, b_{-1}, \ldots\}$. We leave it to the reader to verify that each of these three algebras is simple.

Let $\mathbf{S}$ be an infinite subdirectly irreducible member of $\mathsf{V}(\mathbf{A}')$. Then arguing as in Lemma 6.11 of [2] we conclude that every finite subalgebra of $\mathbf{S}$ is embeddable into a finite large subdirectly irreducible algebra and hence into $\mathbf{Q}'_Z$. From this we conclude that $\mathbf{S}$ satisfies all universal sentences that hold in $\mathbf{Q}'_Z$ and from this it follows that $\mathbf{S}$ is an expansion by term operations of the algebra $\mathbf{S}' = \langle S, 0, \wedge, \cdot, \circ, \leftarrow \rangle$ and that $\langle S, 0, \wedge \rangle$ is a semilattice of height one.

We wish to choose a nonzero element b so that $(0, b)$ lies in the monolith of $\mathbf{S}$. Since $\mathbf{S}$ is an expansion of a height one semilattice then this can be done. We claim that b can be chosen so that it is of the form $x \cdot y$ or $x \circ y$ for some elements x, y from S. Certainly b is in the range of one of the operations $\cdot$, $\circ$ or $\leftarrow$, and if it is of the form $x \leftarrow y$ then it is not hard to show that the element y is also monolith-related to 0 and is in the range of the operation $\circ$ since $y = x \circ (x \cdot y)$ in this case.

We claim that the isomorphism class of $\mathbf{S}$ is determined by whether or not we can choose b as above so that it lies in the range of one of $\cdot$ and $\circ$ but not both. Let us examine in detail the case in which we can choose an element b with $(0, b)$ in the monolith of $\mathbf{S}$ and with b in the range of $\cdot$ but not in the range of $\circ$. As in the proof of Lemma 6.11 of [2] we claim that:

Claim 1 *For each $n \geq 0$, $\mathbf{S}$ has a unique system of elements*

$$\langle c_0, \ldots, c_n, d_0, \ldots, d_{n+1} \rangle$$

satisfying $d_0 = b$, $c_k \cdot d_{k+1} = d_k$, $c_k \circ d_k = d_{k+1}$ and $c_k \leftarrow d_{k+1} = c_k$ for $0 \leq k \leq n$. Furthermore these elements, along with 0, form a subalgebra $\mathbf{S}_n$ of $\mathbf{S}$ and we have that the S_n's form an increasing chain whose union is S.

The proof of this claim is similar to the proof of Claim 2 of 6.11. Our assumption on b not being in the range of $\circ$ is used to conclude that the isomorphism φ between $\mathbf{F}'/\theta$ and $\mathbf{Q}'_n$ established in the proof of Claim 2 must carry b/θ to the element b_0 of $\mathbf{Q}'_n$ since this is the only one of the b_i's in Q'_n which is not in the range of $\circ$.

With this Claim established, it then follows that the algebra $\mathbf{S}$ is isomorphic to $\mathbf{Q}'_\omega$. The case in which we can choose the element b so that it lies in the range of $\circ$ but not in the range of $\cdot$ can be handled similarly to show that in this case the algebra $\mathbf{S}$ is isomorphic to the simple algebra $\mathbf{Q}'_{-\omega}$.

The remaining case to consider is that for all elements b of S with $(0, b)$ in the monolith, b is in the range of $\circ$ if and only if it is in the range of $\cdot$. In this case the following claim can be established:

Claim 2 *For each $n \geq 0$,* $\mathbf{S}$ *has a unique system of elements*

$$\langle c_{-n}, c_{-n+1}, \ldots, c_0, \ldots, c_n, d_{-n}, \ldots, d_0, \ldots, d_{n+1} \rangle$$

satisfying $d_0 = b$, $c_k \cdot d_{k+1} = d_k$, $c_k \circ d_k = d_{k+1}$ and $c_k \leftarrow d_{k+1} = c_k$ for $-n \leq k \leq n$. These elements, along with 0, form a subalgebra $\mathbf{S}_n$ of $\mathbf{S}$ and the collection of the S_n's forms an increasing chain whose union is S.

The verification of this claim is left to the reader as is the proof that with this Claim one can establish that $\mathbf{S}$ is isomorphic to $\mathbf{Q}'_Z$.

4 Concluding Remarks

For an algebra $\mathbf{A}$, the cardinal $\kappa(\mathbf{A})$ defined to be the least cardinal λ such that every subdirectly irreducible algebra in the variety generated by $\mathbf{A}$ has size less than λ, or ∞ if there is no cardinal bound, was introduced and studied in [2]. If we define $\kappa'(\mathbf{A})$ to be the smallest cardinal λ such that every simple algebra in the variety generated by $\mathbf{A}$ has size less than λ, or ∞ if there is no cardinal bound, then what we have established in this article is the existence of a finite algebra of finite type, $\mathbf{A}'$, with $\kappa'(\mathbf{A}') = \aleph_1$.

In the article [3] McKenzie shows that $\kappa(\mathbf{A})$ is not algorithmically computable for finite algebras $\mathbf{A}$ of finite type. His proof is based on a modification of the algebra $\mathbf{A}$ from section 6 of [2]. We expect that the function $\kappa'(\mathbf{A})$ is also not algorithmically computable for finite algebras of finite type but we have not verified this.

In [4] R. Willard introduces a general method for constructing algebras similar to those presented in this paper and in McKenzie's [2]. The reader is encouraged to consult this paper to see how these two examples fit into a general pattern.

References

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