CLASSWORK

1. Differential Equation

A **differential equation** is an equation that contains an unknown function and one or more of it's derivatives.

Example 1. $y' = y$

Example 2. $y'' - 2y' + y = 0$

The order of a differential equation is the higher derivative that occurs in the equation. For instance, example 1 is a **first** order differential equation, while example 2 is a **second** order differential equation.

Definition. A function $f(x)$ is called a solution to a differential equation if the equation is satisfied when $y = f(x)$ and it's derivatives are plugged into the equation.

Question 1. Check that $y = xe^x$ is a solution to example 2.

$$y' = xe^x + e^x$$
$$y'' = xe^x + e^x + e^x = xe^x + 2e^x$$
$$y'' - 2y' + y = xe^x + 2e^x - 2(xe^x + e^x) + e^x = 0$$

Therefore, $y = xe^x$ is a solution to the given differential equation.

As I mentioned last class, we will be interested in solving a specific type of differential equation, called seperable differential equations.

Definition. A **seperable differential equation** is first order differential equation that can be written as

$$\frac{dy}{dx} = g(x) \cdot h(y)$$

It is called seperable because right hand side can be seperated a product between a function of x and a function of y. We solve a seperable differential equation by seperating the x's and the y's,

$$\frac{1}{h(y)} dy = g(x) dx$$

and then integrating

$$\int \frac{1}{h(y)} dy = \int g(x) dx$$

1.1. **Exponential Growth and Decay.** If $y(t)$ is the value of a quantity y at time t and if the rate of change of y with respect to time t is proportional to it's size $y(t)$, then

$$\boxed{\frac{dy}{dx} = k \cdot y} \tag{1}$$

k is a constant and called the **constant of proportionality**.

This differential equation is sometimes called

- The law of natural growth if $k > 0$
- The law of natural decay if $k < 0$.

As we talked about last class, this differential is a seperable differential equation and the solution to Equation 1 is of the form

$$y(t) = y(0)e^{kt} \tag{2}$$

1.2. **Population Growth.** Population growth follows the same differential equation

$$\frac{dP}{dt} = kP,$$

where $k > 0$ and is called the **relative growth rate of population.**

Therefore,

$$P(t) = P(0)e^{kt}$$

where $P(0)$ denotes the population at time 0, and k is the relative growth rate of population.

Question 2. The population of the world was 3040 million in 1960 and 5360 million in 1993. Assuming that the growth rate is proportional to the population size, what is its relative growth rate. Use your answer to predict the world population year 2040. (Measure time in years and let $t = 0$, denote the year 1960.)

$$P(0) = 3040 \implies P(t) = 3040e^{kt}$$

Now we know that $P(33) = 5360$. Then,

$$\begin{aligned} &P(33) = 3040e^{k \cdot 33} \\ &\implies 5360 = 3040e^{33k} \implies e^{33k} = \frac{5360}{3040} \\ &\implies 33k = \ln\left(\frac{5360}{3040}\right) \implies k = \frac{1}{33}\ln\left(\frac{5360}{3040}\right) \\ &\implies k \approx 0.0172 \end{aligned}$$

Therefore,

$$\boxed{P(t) = 3040e^{0.0172t}}$$

To compute the population in year 2040, we plug in $t = 80$ into the above equation to obtain,

$$P(80) = 3040e^{0.0172 \cdot 80} \approx 12035.5$$

1.3. **Radioactive Decay.** Radioactive substance decay spontaneously by emmitting radiation. The mass $m(t)$ is governed by the differential equation,

$$\frac{dm}{dt} = kt,$$

where $k < 0$ and is called the **relative decay rate**.

Then ,

$$m(t) = m(0)e^{kt}$$

Definition. The half life of a radioactive substance is the time required for half of the quantity to decay.

Question 3. The half life of Strontium-90 is 28 days.

a) A sample has a mass of 50 mg initally. Find a formula for the mass remaining after t days.

Initially the sample has mass 50 means that $m(0) = 50$. Therefore,

$$m(t) = 50e^{kt}.$$

The half life is 28 means that at $t = 28$, the mass is half of the original amount i.e

$$m(28) = 25.$$

Now,

$$\begin{aligned} &m(28) = 50e^{k\cdot 28} \\ \implies &e^{28k} = \frac{1}{2} \implies 28k = \frac{1}{28}\ln\left(\frac{1}{2}\right) \\ \implies &k \approx -0.0248 \end{aligned}$$

Therefore,

$$\boxed{m(t) = 50e^{-0.0248t}} \text{ or equivalently } \boxed{m(t) = 50e^{\frac{-\ln 2}{28}t}}$$

b) How long will it take the sample to decay to 20% of its original amount.

20% of the original amount is 10 mg, therefore we want to find t when $m(t) = 10$

So we have,

$$\begin{aligned} &m(t) = 10 \\ \implies &50e^{\frac{-\ln 2}{28}t} = 10 \implies e^{\frac{-\ln 2}{28}t} = \frac{1}{5} \\ \implies &\frac{-\ln 2}{28}t = -\ln(5) \implies t = \frac{\ln 5}{\ln 2}\cdot 28 \approx 65.014 \end{aligned}$$

1.4. **Newton's Law of Cooling.** Newton's law of cooling states that the rate of cooling of an object is proportional to the temperature difference between the object and its surrounding (provided the differnce is not too large).

Let $T(t)$ be the temperature of the object at time t and T_s be its surrounding temperature , then we can formulate Newton's Law of Cooling as a differential equation:

$$\boxed{\frac{dT}{dt} = k(T - T_s),} \tag{3}$$

where k is a constant.

Question 4. Show that the general solution of Equation 3 is

$$T(t) = T_s + (T(0) - T_s)e^{kt}$$

$$\frac{dT}{dt} = k(T - T_s) \implies \frac{dT}{T - T_s} = kdt$$

Now integrating both sides,

$$\int \frac{dT}{T - T_s} = \int kdt$$
$$\implies \ln|T - T_s| = kt + C$$
$$\implies T - T_s = e^{kt+C} = Ae^{kt} \quad \text{(Here we use } A = e^C \text{)}$$
$$\implies T(t) - T_s = Ae^{kt}$$

Plug in $t = 0$ in the above equation. We obtain

$$T(0) - T_s = A$$

Finally we obtain,

$$T(t) - T_s = (T(0) - T_s)e^{kt}$$

Forensics experts use Newton's Law of Cooling to find out when victims of crimes died. They take the temperature of the body when they find it, and by knowing that the average temperature of the human body is 98.6° initially and measuring the room temperature, they can find k and then find t. Why don't you have a go at solving the following mystery.

Case of the Cooling Corpse

It was a dark and stormy night. Holmes and Watson were called by Inspector Lestrade of the police to the scene of the murder. The victim was a wealthy but cruel man. He had many enemies.The most likely suspects are the wife, the business partner, and the butler. Each has an equally strong motive. Each also has an alibi. The wife claims to have spent the entire evening at the theater across town. She was seen leaving the theater at 10 : 30 p.m. and returned home at 11 : 00 p.m., going straight up to her bedroom. Her return was verified by the upstairs maid. The business partner claims to have spent the evening working on papers at the office. His wife and household staff verified that he returned home at 10 : 30 p.m. The butler returned to his quarters above the carriage house at 10 : 05 p.m. and did not leave. This was verified by the other servants.

The body was found in the victim's study. Holmes arrived at the scene at 4 : 30 a.m. The room was unusually warm and stuffy. One of the officers went to open a window. Holmes admonished him to delay that action until he completed his investigation of the crime scene. He instructed Watson to determine the temperature of the body. This was found to be 88.0°.

Holmes questioned the servants as to the room temperature during the evening and learned that the victim liked the room warm and that the temperature in the study was always very near the current 76°. Holmes asked Watson to take the temperature of the body again at the conclusion of the inspection of the scene, two hours after the first reading. It was 85.8°.

Question 5. Based on the information above, find k and the time of death. Find the person that doesnt have an alibi at the time of the murder. That is your murderer.

Measure t in hours and set $t = 0$ at $4:30$am.
Then from the above question we know that

$$T(0) = 88;\ T_s = 76.$$

and also,

$$T(2) = 85.8$$

On the other hand,

$$\begin{aligned} & T(2) - 76 = (88 - 76)e^{k\cdot 2} \\ \implies & 85.8 - 76 = (88 - 76)e^{k\cdot 2} \\ \implies & 9.8 = 12e^{2k} \implies \frac{9.8}{12} = e^{2k} \\ \implies & \ln\left(\frac{9.8}{12}\right) = 2k \implies k = \frac{1}{2}\ln\left(\frac{9.8}{12}\right) \approx -0.1013 \end{aligned}$$

Now the time of death can be obtained by finding when the temperature of the body of is 98.6 i.e. when $T(t) - 98.6$.

$$T(t) - 76 = (88 - 76)e^{-0.1013t}$$
$$\implies 98.6 - 76 = 12e^{-0.1013t} \implies 22.6 = 12e^{-0.1013t} \implies \frac{22.6}{12} = e^{-0.1013t}$$
$$\implies t = \frac{1}{-0.1013} \ln\left(\frac{22.6}{12}\right) \approx -6.25 \text{ hours}$$

Therefore, the time of murder was 6 hours and 15 minutes before 4 : 30 am i.e. the time of the murder is 10 : 15 pm.

The business partner is the only one who does not have an alibi at this time. Therefore, the business partner did it.

Question 6. Solve the initial value problem

$$\frac{dr}{dt} + 2tr = r, \quad r(0) = 5.$$

Note that this differential equation is seperable, because we can rewrite it as

$$\frac{dr}{dt} = \underbrace{r}_{f(r)} \cdot \underbrace{(1-2t)}_{g(t)}$$

So we can rewrite the equation as

$$\frac{dr}{r} = (1-2t)dt$$

Integrating both sides we obtain

$$\int \frac{dr}{r} = \int (1-2t)\, dt$$
$$\implies \ln|r| = t - t^2 + C$$
$$\implies r(t) = Ae^{t-t^2} \quad (\text{Here we use } \ A = e^C\ )$$

To detemine the constant A we use the inital value condition $r(0) = 5$.
Then

$$r(0) = Ae^{0-0^2} \implies 5 = A$$

Therefore

$$\boxed{r(t) = 5e^{t-t^2}}$$