Types A and D quiver representation varieties

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(joint work with Ryan Kinser, Allen Knutson)

Given a quiver Q with vertex set Q_0, arrow set Q_1, and dimension vector $\mathbf{d} : Q_0 \to \mathbb{Z}_{\geq 0}$, there is a *representation space* $\mathrm{rep}_Q(\mathbf{d}) := \prod_{a \in Q_1} \mathrm{Mat}(\mathbf{d}(ta), \mathbf{d}(ha))$, where ta and ha denote the tail and head vertices of the arrow a, and $\mathrm{Mat}(m, n)$ denotes the space of $m \times n$ matrices with entries in a field K. The product of general linear groups $\mathrm{GL}(\mathbf{d}) := \prod_{z \in Q_0} \mathrm{GL}_{\mathbf{d}(z)}(K)$ acts on $\mathrm{rep}_Q(\mathbf{d})$ on the right by conjugation, that is, $(V_a)_{a \in Q_1} \cdot (g_z)_{z \in Q_0} := (g_{ta}^{-1} V_a g_{ha})_{a \in Q_1}$, for $(V_a)_{a \in Q_1} \in \mathrm{rep}_Q(\mathbf{d})$, and $(g_z)_{z \in Q_0} \in \mathrm{GL}(\mathbf{d})$. The closure of a $\mathrm{GL}(\mathbf{d})$ orbit in $\mathrm{rep}_Q(\mathbf{d})$ is called a *quiver locus.*

Motivations for the study of quiver loci come from various disciplines including algebraic geometry, commutative algebra, representation theory, and algebraic combinatorics. For example, through the study of quiver loci, one encounters well-known ideals from commutative algebra, including classical determinantal ideals and defining ideals of varieties of complexes. The primary motivation for the work discussed herein is from algebraic geometry, through study of *degeneracy loci*: given a non-singular algebraic variety X and a map of vector bundles $\phi : V \to W$ on X, there is a degeneracy locus $\Omega_r := \{x \in X \mid \text{rank } \phi_x \leq r\}$, where $\phi_x : V_x \to W_x$ is the induced map on fibers. The locus Ω_r is a closed subvariety of X, defined locally by the vanishing of minors of a matrix. When ϕ is sufficiently general, an expression for its fundamental class in the cohomology ring of X is given by the Giambelli-Thom-Porteous formula. A. Buch and W. Fulton generalized this to sequences of vector bundle maps $V_1 \to V_2 \to \cdots \to V_n$ in [6]. Related formulas were subsequently produced in works such as [6, 2, 8, 5, 13, 18, 15, 3, 7].

The problem of producing formulas for degeneracy loci is closely related to that of finding formulas for *multidegrees* of associated quiver loci (see [13]). Indeed, in [13], A. Knutson, E. Miller, and M. Shimozono produced multiple formulas for the multidegrees and *K-polynomials* of quiver loci of equioriented type A quivers (i.e. all arrows point in the same direction). One important ingredient in this work was the *Zelevinsky map*, which identifies an equioriented type A quiver locus with an open subvariety of a Schubert variety [19, 14], thereby allowing for importation of results from Schubert calculus to the equioriented type A quiver setting.

In joint work with R. Kinser and A. Knutson [10], we generalized three of Knutson, Miller, and Shimozono's formulas from [13] to all type A orientations. Our work also generalized or recovered formulas from [16, 4, 7]. Our main result was a proof of A. Buch and R. Rimányi's conjectured K-theoretic component formula from [7]. In analogy with the equioriented setting, an explicit connection to Schubert varieties (from [11]) was important to our work.

This extended abstract focuses on algebro-geometric results on type A quiver loci important to the proofs of the formulas in [10], as well as analogs of some of these algebro-geometric results for type D loci. The latter part is based on recent joint work with Kinser [12].

1. Type A

In this section we discuss some geometric results in type A, including the bipartite Zelevinsky map from [11], and a degeneration from [10] which was central to the proof of the K-theoretic type A quiver component formula.

1.1. **Bipartite Zelevinsky map.** Let Q be a bipartite (i.e. alternating) type A quiver and $\mathbf{d}$ a dimension vector for Q. There is an associated general linear group $\mathrm{GL}_d(K)$, parabolic subgroup $P \subseteq \mathrm{GL}_d(K)$, opposite Schubert cell $Y \subseteq P\backslash\mathrm{GL}_d$, and closed immersion $\zeta : \mathrm{rep}_Q(\mathbf{d}) \to Y$ which restricts to an isomorphism from each quiver locus in $\mathrm{rep}_Q(\mathbf{d})$ to a Schubert variety intersected with Y [11]. We refer to ζ as the *bipartite Zelevinsky map*. For example, the type A_5 representation

$$K^{d_5} \xrightarrow{D} K^{d_4} \xleftarrow{C} K^{d_3} \xrightarrow{B} K^{d_2} \xleftarrow{A} K^{d_1}$$

maps, via ζ, to the block matrix on the left below:

$$\begin{bmatrix} 0 & A & 1 & 0 & 0 \\ C & B & 0 & 1 & 0 \\ D & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} \in \begin{bmatrix} * & * & 1 & 0 & 0 \\ * & * & 0 & 1 & 0 \\ * & * & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} \cong Y.$$

Note that 1s denote identity matrices of appropriate sizes, and the matrix on the right denotes the space of all matrices of the given form (i.e. with arbitrary elements of K in the locations with stars). The ranks of the matrices

$$A, B, C, D, \begin{bmatrix} A \\ B \end{bmatrix}, \begin{bmatrix} C & B \end{bmatrix}, \begin{bmatrix} C \\ D \end{bmatrix}, \begin{bmatrix} 0 & A \\ C & B \end{bmatrix}, \begin{bmatrix} C & B \\ D & 0 \end{bmatrix}, \begin{bmatrix} 0 & A \\ C & B \\ D & 0 \end{bmatrix}$$

characterize the points in the $\mathrm{GL}(\mathbf{d})$ orbit containing (D, C, B, A). This list of ranks is equivalent to a list of ranks of certain North-West justified submatrices of $\zeta(D, C, B, A)$. Using this, one can show that the closure of the orbit through (D, C, B, A) is isomorphic, via ζ, to a Schubert variety intersected with Y.

The bipartite Zelevinsky map is useful beyond the bipartite setting because of the following: given a type A quiver locus $\Omega \subseteq \mathrm{rep}_Q(\mathbf{d})$ (for Q of arbitrary orientation), there is as associated bipartite type A quiver locus $\widetilde{\Omega}$ and product of general linear groups G^* such that $\Omega \times G^*$ is isomorphic to an open subvariety of $\widetilde{\Omega}$ [11]. It follows from this result, the bipartite Zelevinsky map, and [9, Lemma A.4] that each type A quiver locus is isomorphic, up to a smooth factor, to an open subvariety of a Schubert variety. This gives a uniform way of obtaining results such as type A quiver loci are normal and Cohen-Macaulay with rational singularities (also proved earlier via other methods by G. Bobiński and G. Zwara [1]); quiver locus containment is governed by Bruhat order on the symmetric group; there is a Frobenius splitting (in positive characteristic) of each representation space of a type A quiver that compatibly splits all quiver loci; certain generalized determinantal ideals are prime, and scheme-theoretically define type A quiver loci (C. Riedtmann and G. Zwara [17] also obtained this result via other methods).

1.2. **Degenerations.** Certain degenerations of bipartite type A quiver loci are important in the proofs of the component formulas found in [10]. To describe these degenerations (also given in [10]), we start with a general set-up: let G be an algebraic group over K, $H \leq G$ a closed subgroup, X a G-variety, and $Y \subseteq X$ an H-stable closed subvariety. Let $\mu : K^\times \to G$ be a group homomorphism and consider the right action of $K^\times$ on G by $g \cdot t = \mu(t^{-1})g\mu(t)$ and the right action of $K^\times$ on X by $x \cdot t = x \cdot \mu(t)$. With this set-up, we get two families $\widetilde{H}$ and $\widetilde{Y}$ over $\mathbb{A}^1 - \{0\}$ where the fiber $H \cdot t$ in the first family is a subgroup which acts on the fiber $Y \cdot t$ in the second family.

In our case, X is a bipartite type A quiver representation space $\text{rep}_Q(\mathbf{d})$, $Y \subseteq X$ is a quiver locus, $G = \text{GL}(\mathbf{d}) \times \text{GL}(\mathbf{d})$, and $H = \text{GL}(\mathbf{d})^\Delta$, a copy of $\text{GL}(\mathbf{d})$ embedded diagonally in G. The action of G on X is a conjugation action where if $((g_z, h_z))_{z \in Q_0} \in G$ then g_z (respectively h_z) acts on the map over the arrow to the left of z (respectively to the right of z). The induced action of H on X is then the usual action of $\text{GL}(\mathbf{d})$ on $\text{rep}_Q(\mathbf{d})$. Letting $\rho_z(t)$ denote the $\mathbf{d}(z) \times \mathbf{d}(z)$ diagonal matrix with $t, t^2, \dots, t^{\mathbf{d}(z)}$ down the diagonal, the homomorphism μ is defined by $t \mapsto (\ (\rho_z(t^{-1}), \rho_z(t))\ )_{z \in Q_0}$. The families $\widetilde{H}$ and $\widetilde{Y}$ extend to flat families over $\mathbb{A}^1$ and the special fiber of the first family acts on the special fiber of the second family. From this, one deduces that a bipartite type A quiver locus degenerates to a union of products of matrix Schubert varieties, up to radical. One can further prove that this degeneration is reduced. See [10] for details. Similar degenerations appeared previously in the equioriented setting in [13].

2. Type D

In recent joint work with Kinser [12], we obtain results in type D which are analogous to the type A results from [11]. Indeed, we unify aspects of the equivariant geometry of three classes of varieties: type D quiver representation varieties, double Grassmannians $Gr(a, n) \times Gr(b, n)$, and symmetric varieties $GL(a + b)/(GL(a) \times GL(b))$. In particular, we translate results about singularities of orbit closures, combinatorics of orbit closure containment, and torus equivariant K-theory between these three families. This is accomplished by producing explicit embeddings of homogeneous fiber bundles over type D quiver representation spaces into symmetric varieties. Immediate consequences of these embeddings, together with results on symmetric varieties, include type D quiver loci are normal and Cohen-Macaulay with rational singularities (recovering work from [1] obtained by other methods); the poset of orbit closures in a type D representation space (and also the poset of diagonal B-orbit closures in a double Grassmannian) is isomorphic to a subposet of a poset of *clans*, which are involutions in the symmetric group with signed fixed points.

A next step in the investigation of type D quiver loci is to make use of the explicit embeddings in [12] to help produce formulas for multidegrees and K-polynomials, in analogy with what was done in type A.

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