AN EXPANDED VERSION OF '*TWO WEIGHT INEQUALITY FOR THE HILBERT TRANSFORM: A REAL VARIABLE CHARACTERIZATION*'

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Abstract. This paper is an expanded version of the paper '*Two Weight Inequality for the Hilbert Transform: A Real Variable Characterization*' by M. L. Lacey, E. T. Sawyer, C.-Y. Shen and I. Uriarte-Tuero. The arguments here are intended to be read by nonexperts, so there is more background, longer proofs, and a slight reorganization of the overall plan. All of the arguments here are due to the four authors mentioned above, but any errors, omissions and/or confusion introduced in this expanded version are due to this author alone. Let σ and ω be locally finite positive Borel measures on $\mathbb{R}$ with no common point masses. We show boundedness of the Hilbert transform $H_\sigma f \equiv H(f\sigma)$ from $L^2(\sigma)$ to $L^2(\omega)$ is equivalent to the $\mathcal{A}_2$ condition

$$|I|_\sigma \int_I \left(\frac{|I|}{|I|+|x-x_I|}\right)^2 d\omega(x) + |I|_\omega \int_I \left(\frac{|I|}{|I|+|x-x_I|}\right)^2 d\sigma(x) \leq C|I|^2,$$

and the two *indicator/interval* testing conditions,

$$\begin{aligned} \int_I (H_\sigma \mathbf{1}_E)^2 \, d\omega &\leq C|I|_\sigma, \\ \int_I (H_\omega \mathbf{1}_E)^2 \, d\sigma &\leq C|I|_\omega, \end{aligned}$$

holding for all intervals I and compact subsets E of I (note that E does not appear on the right side of the testing conditions). In particular, H_σ is bounded if and only if both H_σ and its dual H_ω are weak type $(2,2)$.

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1. Introduction

In this paper we give a proof with expanded details, and additional background, of the real variable characterization of the two weight inequality for the Hilbert transform given in [LaSaShUr2] by M. L. Lacey, E. T. Sawyer, C.-Y. Shen and I. Uriarte-Tuero. There is also a slight reorganization of the proof as given in [LaSaShUr2]. All of the arguments are due to Lacey, Sawyer, Shen and Uriarte-Tuero, but any errors, omissions and/or confusion introduced into this expanded version are due to this author alone.

Let $H\nu(x) = \int_{\mathbb{R}} \frac{d\nu(y)}{y-x}$ be the Hilbert transform of the measure ν. The principal value associated with this definition need not exist in general, so we always understand that there is a fixed standard truncation of the kernel in place here. Given weights (i.e. locally bounded positive Borel measures) σ and ω on the real line $\mathbb{R}$ with no common point masses, we characterize the following *two weight norm inequality* for the Hilbert transform,

$$\int_{\mathbb{R}} |H(f\sigma)|^2 \omega \leq \mathfrak{N} \int_{\mathbb{R}} |f|^2 \sigma, \qquad f \in L^2(\sigma), \tag{1.1}$$

uniform over all standard truncations of the Hilbert transform kernel. A question raised in [Vol], which we refer to as the *NTV conjecture*, is whether or not (1.1) is equivalent to the following necessary conditions (see [NTV4] and [LaSaUr1] for the necessity of $\mathcal{A}_2 < \infty$),

$$\begin{aligned} \mathsf{P}(I,\sigma)\frac{|I|_\omega}{|I|} &\leq \mathcal{A}_2, \quad \frac{|I|_\sigma}{|I|}\mathsf{P}(I,\omega) \leq \mathcal{A}_2, \\ \int_I |H(\mathbf{1}_I\sigma)|^2 \omega &\leq \mathfrak{T}^2 |I|_\sigma, \quad \int_I |H(\mathbf{1}_I\omega)|^2 \sigma \leq (\mathfrak{T}^*)^2 |I|_\omega, \end{aligned} \tag{1.2}$$

called the $\mathcal{A}_2$ condition and the two interval testing conditions. A weaker conjecture is the *indicator/interval NTV conjecture* in which the interval testing conditions are replaced by the indicator/interval testing conditions

$$\int_I |H(\mathbf{1}_E\sigma)|^2 \omega \leq \mathfrak{T}_{ind}^2 |I|_\sigma, \quad \int_I |H(\mathbf{1}_E\omega)|^2 \sigma \leq (\mathfrak{T}_{ind}^*)^2 |I|_\omega,$$

for all E compact $\subset I$ interval. Note that E does not appear on the right side of the inequalities, and that for a *positive* operator H, the indicator/interval and interval testing conditions are the same. It is an elementary exercise to establish the equivalence of the indicator/interval testing condition with

$$\int_I |H(f\mathbf{1}_I\sigma)|^2\,\omega \le \mathfrak{T}_{ind}^2\,|I|_\sigma\,,\quad \int_I |H(f\mathbf{1}_I\omega)|^2\,\sigma \le (\mathfrak{T}_{ind}^*)^2\,|I|_\omega\,, \tag{1.3}$$

for all intervals I and functions f with $|f|\le 1$ (see the appendix).

In this paper we prove the *indicator/interval NTV conjecture.*

Theorem 1. *Let σ and ω be locally finite positive Borel measures on the real line $\mathbb{R}$ with no common point masses. The best constants $\mathfrak{N}$, $\mathcal{A}_2$, $\mathfrak{T}_{ind}$, and $\mathfrak{T}_{ind}^*$ in (1.1), (1.2) and (1.3) satisfy*

$$\mathfrak{N}\approx\sqrt{\mathcal{A}_2}+\mathfrak{T}_{ind}+\mathfrak{T}_{ind}^*.$$

Since the constant on the right side above arises repeatedly throughout the paper, we set $\mathfrak{NTV}_{ind}=\sqrt{\mathcal{A}_2}+\mathfrak{T}_{ind}+\mathfrak{T}_{ind}^*$. We also set $\mathfrak{NTV}=\sqrt{\mathcal{A}_2}+\mathfrak{T}+\mathfrak{T}^*$. Here is an operator theoretic consequence of the theorem.

Corollary 1.4. *Let σ and ω be locally finite positive Borel measures on the real line $\mathbb{R}$ with no common point masses. Denote by $\mathfrak{W}$ and $\mathfrak{W}^*$ the weak type $(2,2)$ norms of H_σ and H_ω respectively, i.e.*

$$\begin{aligned}\lambda\ \left|\{|H(f\sigma)|>\lambda\}\right|_\omega^{\frac12} &\le \mathfrak{W}\ \|f\|_{L^2(\sigma)}\,, & f\in L^2(\sigma)\,,\\ \lambda\ \left|\{|H(g\omega)|>\lambda\}\right|_\sigma^{\frac12} &\le \mathfrak{W}^*\ \|g\|_{L^2(\omega)}\,, & g\in L^2(\omega)\,.\end{aligned}$$

Then

$$\mathfrak{N}\approx\mathfrak{W}+\mathfrak{W}^*.$$

The corollary follows from the theorem since duality and the theory of Lorentz spaces give $\mathfrak{T}_{ind}\le\mathfrak{W}^*$ and $\mathfrak{T}_{ind}^*\le\mathfrak{W}$; while $\sqrt{\mathcal{A}_2}\lesssim\mathfrak{W}+\mathfrak{W}^*$ is evident from the proof of $\sqrt{\mathcal{A}_2}\lesssim\mathfrak{N}$ in [LaSaUr].

Finally, current interest in the two weight problem for the Hilbert transform arises from its natural occurrence in questions related to operator theory [NiTr], spectral theory, model spaces [NaVo], and analytic function spaces [2], among others.

1.1. A brief history of the problem.

The two weight norm inequality (1.1) for the Hilbert transform became recognized as a difficult problem shortly after the classical one-weight problem was solved in 1973 by R. Hunt, B. Muckenhoupt and R.L. Wheeden [HuMuWh]. While success in related two weight problems for positive operators came relatively quickly in the early 1980's in [Saw1] and [Saw3], the case of singular integrals remained mysterious for some time. Progress in a different direction was achieved by David and Journé in 1984 when they solved in [DaJo] the norm inequality for general Calderon-Zygmund operators, but with Lebesgue measure as the weights. It was not until the late 1990's and early this millenium that significant inroads were made in the singular two weight problem by F. Nazarov, S. Treil and A. Volberg using their recently developed techniques for harmonic analysis on nondoubling spaces, see e.g. [NaVo], [NTV1] and [NTV2].

This effort culminated in the beautiful arguments in the 2004 preprint [NTV4] and 2003 CBMS book [Vol], in which NTV followed the form of characterizations in [Saw1], [Saw3] and [DaJo], by showing that (1.1) was implied by (1.2) if certain side conditions were imposed, namely the pivotal condition,

$$\sum_{n=1}^{\infty}\left|J_n\right|_{\omega}\mathsf{P}\left(J_n,\mathbf{1}_I\sigma\right)^2\lesssim\left|I\right|_{\sigma}\,,\qquad\overset{\cdot}{\bigcup}_{n=1}^{\infty}J_n\subset I,$$

and its dual. The proof analyzed the bilinear form $\left\langle H_{\sigma}f,g\right\rangle_{\omega}$ by expanding f and g in random Haar bases,

$$\left\langle H_{\sigma}f,g\right\rangle_{\omega}=\sum_{I,J}\left\langle H_{\sigma}\bigtriangleup_I^{\sigma}f,\bigtriangleup_J^{\omega}g\right\rangle_{\omega},$$

splitting the forms into upper and lower and diagonal forms according to the relative lengths of the dyadic intervals I and J, and then using a new corona argument that involved stopping times defined with respect to the pivotal condition.

In [LaSaUr], three of us showed that the pivotal conditions were not necessary for (1.1), and weakened these side conditions extensively, but were not able to completely eliminate them. Also in that paper, the concept of the energy

$$\mathsf{E}\left(J,\omega\right)=\left\{\frac{1}{\left|J\right|_{\omega}}\int_J\left(\mathbb{E}_J^{\omega\left(dx'\right)}\frac{x-x'}{\left|J\right|}\right)^2d\omega\left(x\right)\right\}^{\frac{1}{2}},$$

of a weight ω on an interval J was introduced, and the energy versions of the pivotal conditions were shown to be necessary for 1.1, namely

$$\sum_{n=1}^{\infty}\left|J_n\right|_{\omega}\mathsf{E}\left(J_n,\omega\right)^2\mathsf{P}\left(J_n,\mathbf{1}_I\sigma\right)^2\lesssim\left(\mathfrak{NTV}\right)\left|I\right|_{\sigma}\,,\qquad\overset{\cdot}{\bigcup}_{n=1}^{\infty}J_n\subset I,$$

and its dual condition. However, superadditivity of the functional $J\to\left|J\right|_{\omega}$ appearing in the pivotal condition was a crucial property for the NTV proof strategy, which involved a clever estimate of off-diagonal terms in the Haar expansion of the bilinear form $\left\langle H_{\sigma}f,g\right\rangle_{\omega}$. Unfortunately, this crucial property fails for the corresponding functional $J\to\left|J\right|_{\omega}\mathsf{E}\left(J,\omega\right)^2$ appearing in the energy condition, and the sufficiency proof stalled dued to inadequate control of the energy stopping time coronas.

Both the pivotal and energy stopping times used in [NTV4] and [LaSaUr] depend only on the weights ω and σ, and not on the functions f and g involved in the form. In [LaSaShUr] the current authors introduced Calderón-Zygmund stopping times into the argument, which had been previously used for maximal truncations of Hilbert transform in [LaSaUr1], and which depend on the averages of the moduli of the functions involved. But the failure of the weights ω and σ to be doubling presented a formidable obstacle in [LaSaShUr] just as in [LaSaUr1], and moreover, this approach highlighted the fact that the splitting of the form $\left\langle H_{\sigma}f,g\right\rangle_{\omega}$ according to relative lengths of the intervals I and J might *not* be a bounded operation in general, hence dooming this splitting from the start (see [LaSaShUr] for more detail on the question of bounding the split forms, which remains open at the time of this writing).

1.1.1. *Circumventing the obstacles.* The difficulties mentioned above are circumvented in the present paper by introducing a new splitting of the bilinear form, followed by a careful analysis of the extremal functions that fail both the energy and Calderón-Zygmund stopping time methodology. The new splitting is the *parallel corona* splitting that involves defining upper and lower and diagonal forms relative to the tree of Calderón-Zygmund stopping time intervals, rather than the full tree of dyadic intervals. Recall that the enemy of Calderón-Zygmund stopping times is degeneracy of the doubling property, while the enemy of energy stopping times is degeneracy of the energy functional (since nondegenerate doubling implies nondegenerate energy, it is really the failure of doubling in both weights that is the common enemy). A series of three reductions are then performed with Calderón-Zygmund and energy parallel coronas to identify the extremal functions that fail to yield to the standard analyses, such as certain bounded functions, and functions of *minimal bounded fluctuation* with energy control. In the end, the standard NTV methodology is decisive when used on these extremal functions with very special structure.

1.2. **The parallel corona decomposition.** The main construction in our proof of Theorem 1 is the following parallel corona decomposition, which improves the decomposition according to interval side length that has been used in all previous papers, in particular in [NTV4], [LaSaUr] and [LaSaShUr]. Let $\mathcal{D}^\sigma$ and $\mathcal{D}^\omega$ be an r-good pair of grids, and let $\{h_I^\sigma\}_{I\in\mathcal{D}^\sigma}$ and $\{h_J^\omega\}_{J\in\mathcal{D}^\omega}$ be the corresponding Haar bases, so that

$$\begin{aligned} f &= \sum_{I\in\mathcal{D}^\sigma} \triangle_I^\sigma f = \sum_{I\in\mathcal{D}^\sigma} \langle f, h_I^\sigma\rangle \ h_I^\sigma = \sum_{I\in\mathcal{D}^\sigma} \widehat{f}(I)\ h_I^\sigma, \\ g &= \sum_{J\in\mathcal{D}^\omega} \triangle_J^\omega g = \sum_{J\in\mathcal{D}^\omega} \langle g, h_J^\omega\rangle \ h_J^\omega = \sum_{J\in\mathcal{D}^\omega} \widehat{g}(J)\ h_J^\omega, \end{aligned}$$

where the appropriate grid is understood in the notation $\widehat{f}(I)$ and $\widehat{g}(J)$. It is convenient to define $H_\sigma f \equiv H(f\sigma)$ so that the dual operator H_σ^* is H_ω: $\langle H_\sigma f, g\rangle_\omega = \langle f, H_\omega g\rangle_\sigma$.

Inequality (1.1) is equivalent to boundedness of the bilinear form

$$\mathcal{H}(f,g) \equiv \langle H_\sigma(f), g\rangle_\omega = \sum_{I\in\mathcal{D}^\sigma \text{ and } J\in\mathcal{D}^\omega} \langle H_\sigma(\triangle_I^\sigma f), \triangle_J^\omega g\rangle_\omega$$

on $L^2(\sigma)\times L^2(\omega)$, i.e.

$$|\mathcal{H}(f,g)| \leq \mathfrak{N}\, \|f\|_{L^2(\sigma)} \|g\|_{L^2(\omega)}.$$

Virtually all attacks on the two weight inequality (1.1) to date have proceeded by first splitting the bilinear form $\mathcal{H}$ into three natural forms determined by the relative size of the intervals I and J in the inner product:

$$\begin{aligned} \mathcal{H} &= \mathcal{H}_{lower} + \mathcal{H}_{diagonal} + \mathcal{H}_{upper}; \\ \mathcal{H}_{lower}(f,g) &\equiv \sum_{\substack{I\in\mathcal{D}^\sigma \text{ and } J\in\mathcal{D}^\omega \\ |J|<2^{-r}|I|}} \langle H_\sigma(\triangle_I^\sigma f), \triangle_J^\omega g\rangle_\omega, \\ \mathcal{H}_{diagonal}(f,g) &\equiv \sum_{\substack{I\in\mathcal{D}^\sigma \text{ and } J\in\mathcal{D}^\omega \\ 2^{-r}|I|\leq|J|\leq 2^r|I|}} \langle H_\sigma(\triangle_I^\sigma f), \triangle_J^\omega g\rangle_\omega, \\ \mathcal{H}_{upper}(f,g) &\equiv \sum_{\substack{I\in\mathcal{D}^\sigma \text{ and } J\in\mathcal{D}^\omega \\ |J|>2^r|I|}} \langle H_\sigma(\triangle_I^\sigma f), \triangle_J^\omega g\rangle_\omega, \end{aligned} \tag{1.5}$$

and then continuing to establish boundedness of each of these three forms. Now the boundedness of the diagonal form $\mathcal{H}_{diagonal}$ is an automatic consequence of that of $\mathcal{H}$ since it is shown in [NTV4] that

$$\left|\mathcal{H}_{diagonal}(f,g)\right| \lesssim (\mathfrak{NTV}) \|f\|_{L^2(\sigma)} \|g\|_{L^2(\omega)} \lesssim \mathfrak{N} \|f\|_{L^2(\sigma)} \|g\|_{L^2(\omega)}.$$

However, it is *not* known if the boundedness of $\mathcal{H}_{lower}$ and $\mathcal{H}_{upper}$ follow from that of $\mathcal{H}$, which places in jeopardy the entire method of attack based on the splitting (1.5) of the form $\mathcal{H}$. See [LaSaShUr] for a detailed discussion of these matters.

In order to improve on the splitting in (1.5), we introduce stopping trees $\mathcal{F}$ and $\mathcal{G}$ for the functions $f \in L^2(\sigma)$ and $g \in L^2(\omega)$. Let $\mathcal{F}$ (respectively $\mathcal{G}$) be a collection of Calderón-Zygmund stopping intervals for f (respectively g), and let $\mathcal{D}^\sigma = \bigcup_{F\in\mathcal{F}} \mathcal{C}_F$ (respectively $\mathcal{D}^\omega = \bigcup_{G\in\mathcal{G}} \mathcal{C}_G$) be the associated corona decomposition of the dyadic grid $\mathcal{D}^\sigma$ (respectively $\mathcal{D}^\omega$). For $I \in \mathcal{D}^\sigma$ let $\pi_{\mathcal{D}^\sigma} I$ be the $\mathcal{D}^\sigma$-parent of I in the grid $\mathcal{D}^\sigma$, and let $\pi_{\mathcal{F}} I$ be the smallest member of $\mathcal{F}$ that contains I. For $F, F' \in \mathcal{F}$, we say that F' is an $\mathcal{F}$-child of F if $\pi_{\mathcal{F}}(\pi_{\mathcal{D}^\sigma} F') = F$ (it could be that $F = \pi_{\mathcal{D}^\sigma} F'$), and we denote by $\mathfrak{C}_{\mathcal{F}}(F)$ the set of $\mathcal{F}$-children of F. For $F \in \mathcal{F}$, define the projection $\mathsf{P}^\sigma_{\mathcal{C}_F}$ onto the linear span of the Haar functions $\{h^\sigma_I\}_{I\in\mathcal{C}_F}$ by

$$\mathsf{P}^\sigma_{\mathcal{C}_F} f = \sum_{I\in\mathcal{C}_F} \triangle^\sigma_I f = \sum_{I\in\mathcal{C}_F} \langle f, h^\sigma_I\rangle_\sigma h^\sigma_I.$$

The standard properties of these projections are

$$f = \sum_{F\in\mathcal{F}} \mathsf{P}^\sigma_{\mathcal{C}_F} f, \qquad \int \left(\mathsf{P}^\sigma_{\mathcal{C}_F} f\right)\sigma = 0, \qquad \|f\|^2_{L^2(\sigma)} = \sum_{F\in\mathcal{F}} \left\|\mathsf{P}^\sigma_{\mathcal{C}_F} f\right\|^2_{L^2(\sigma)}.$$

There are similar definitions and formulas for the tree $\mathcal{G}$ and grid $\mathcal{D}^\omega$.

Remark 1.6. *The stopping intervals $\mathcal{F}$ live in the full dyadic grid $\mathcal{D}^\sigma$, while the intervals $I \in \mathcal{C}_F$ are restricted to the* good *subgrid $\mathcal{D}^\sigma_{good}$ as defined in Subsection 3.3 below. It is important to observe that the arguments used in this paper never appeal to a 'good' property for stopping intervals, only for intervals in the Haar support of f. A similar remark applies to $\mathcal{G}$ and the Haar support of g.*

We now consider the following *parallel corona splitting* of the inner product $\langle H(f\sigma), g\rangle_\omega$ that involves the projections $\mathsf{P}^\sigma_{\mathcal{C}_F}$ acting on f and the projections $\mathsf{P}^\omega_{\mathcal{C}_G}$ acting on g. These forms are no longer linear in f and g as the 'cut' is determined by the coronas $\mathcal{C}_F$ and $\mathcal{C}_G$, which depend on f and g. We have

$$(1.7)\quad \begin{aligned} \langle H_\sigma f, g\rangle_\omega &= \sum_{(F,G)\in\mathcal{F}\times\mathcal{G}} \left\langle H_\sigma\left(\mathsf{P}^\sigma_{\mathcal{C}_F} f\right), \left(\mathsf{P}^\omega_{\mathcal{C}_G} g\right)\right\rangle_\omega \\ &= \left\{\sum_{(F,G)\in\mathsf{Near}(\mathcal{F}\times\mathcal{G})} + \sum_{(F,G)\in\mathsf{Disjoint}(\mathcal{F}\times\mathcal{G})} + \sum_{(F,G)\in\mathsf{Far}(\mathcal{F}\times\mathcal{G})}\right\} \\ &\qquad \times \left\langle H_\sigma\left(\mathsf{P}^\sigma_{\mathcal{C}_F} f\right), \left(\mathsf{P}^\omega_{\mathcal{C}_G} g\right)\right\rangle_\omega \\ &\equiv \mathsf{H}_{near}(f,g) + \mathsf{H}_{disjoint}(f,g) + \mathsf{H}_{far}(f,g). \end{aligned}$$

Here $\mathsf{Near}(\mathcal{F}\times\mathcal{G})$ is the set of pairs $(F,G) \in \mathcal{F}\times\mathcal{G}$ such that G is the *minimal* interval in $\mathcal{G}$ that contains F, or F is the *minimal* interval in $\mathcal{F}$ that contains G, more precisely: either

$$F \subset G \text{ and there is no } G_1 \in \mathcal{G} \setminus \{G\} \text{ with } F \subset G_1 \subset G,$$

or

$$G \subset F \text{ and there is no } F_1 \in \mathcal{F} \setminus \{F\} \text{ with } G \subset F_1 \subset F.$$

The set $\mathsf{Disjoint}(\mathcal{F} \times \mathcal{G})$ is the set of pairs $(F,G) \in \mathcal{F} \times \mathcal{G}$ such that $F \cap G = \emptyset$. The set $\mathsf{Far}(\mathcal{F} \times \mathcal{G})$ is the complement of $\mathsf{Near}(\mathcal{F} \times \mathcal{G}) \cup \mathsf{Disjoint}(\mathcal{F} \times \mathcal{G})$ in $\mathcal{F} \times \mathcal{G}$:

$$\mathsf{Far}(\mathcal{F} \times \mathcal{G}) = \mathcal{F} \times \mathcal{G} \setminus \{\mathsf{Near}(\mathcal{F} \times \mathcal{G}) \cup \mathsf{Disjoint}(\mathcal{F} \times \mathcal{G})\}.$$

The parallel corona splitting (1.7) is somewhat analogous to the splitting (1.5) except that the stopping intervals at the top of the corona blocks are used in place of the individual intervals within the coronas to determine the 'cut'. It is this feature that permits our characterization of the two weight inequality (1.1) in terms of $\mathcal{A}_2$ and indicator/interval testing conditions.

Before moving on, it is convenient to introduce a corona decomposition that uses stopping data more general in scope than the Calderón-Zygmund data.

1.3. **General stopping data.** Our general definition of stopping data will use a positive constant $C_0 \geq 4$.

Definition 1.8. *Suppose we are given a positive constant $C_0 \geq 4$, a subset $\mathcal{F}$ of the dyadic grid $\mathcal{D}^\sigma$ (called the stopping times), and a corresponding sequence $\alpha_{\mathcal{F}} \equiv \{\alpha_{\mathcal{F}}(F)\}_{F\in\mathcal{F}}$ of nonnegative numbers $\alpha_{\mathcal{F}}(F) \geq 0$ (called the stopping data). Let $(\mathcal{F}, \prec, \pi_{\mathcal{F}})$ be the tree structure on $\mathcal{F}$ inherited from $\mathcal{D}^\sigma$, and for each $F \in \mathcal{F}$ denote by $\mathcal{C}_F = \{I \in \mathcal{D}^\sigma : \pi_{\mathcal{F}} I = F\}$ the corona associated with F:*

$$\mathcal{C}_F = \{I \in \mathcal{D}^\sigma : I \subset F \text{ and } I \not\subset F' \text{ for any } F' \prec F\}.$$

We say the triple $(C_0, \mathcal{F}, \alpha_{\mathcal{F}})$ constitutes stopping data *for a function $f \in L^1_{loc}(\sigma)$ if*

1. $\mathbb{E}^\sigma_I |f| \leq \alpha_{\mathcal{F}}(F)$ *for all* $I \in \mathcal{C}_F$ *and* $F \in \mathcal{F}$,
2. $\sum_{F' \preceq F} |F'|_\sigma \leq C_0 |F|_\sigma$ *for all* $F \in \mathcal{F}$,
3. $\sum_{F\in\mathcal{F}} \alpha_{\mathcal{F}}(F)^2 |F|_\sigma \leq C_0^2 \|f\|^2_{L^2(\sigma)}$,
4. $\alpha_{\mathcal{F}}(F) \leq \alpha_{\mathcal{F}}(F')$ *whenever* $F', F \in \mathcal{F}$ *with* $F' \subset F$.

Definition 1.9. *If $(C_0, \mathcal{F}, \alpha_{\mathcal{F}})$ constitutes (general)* stopping data *for a function $f \in L^1_{loc}(\sigma)$, we refer to the othogonal decomposition*

$$f = \sum_{F\in\mathcal{F}} \mathsf{P}^\sigma_{\mathcal{C}_F} f; \qquad \mathsf{P}^\sigma_{\mathcal{C}_F} f \equiv \sum_{I\in\mathcal{C}_F} \triangle^\sigma_I f,$$

as the (general) corona decomposition *of f associated with the stopping times $\mathcal{F}$.*

Property (1) says that $\alpha_{\mathcal{F}}(F)$ bounds the averages of f in the corona $\mathcal{C}_F$, and property (2) says that the intervals at the tops of the coronas satisfy a Carleson condition relative to the weight σ. Note that a standard 'maximal interval' argument extends the Carleson condition in property (2) to the inequality

$$\sum_{F'\in\mathcal{F}:\ F'\subset A} |F'|_\sigma \leq C_0 |A|_\sigma \text{ for all open sets } A \subset \mathbb{R}.$$

Property (3) says the sequence of functions $\{\alpha_{\mathcal{F}}(F) \mathbf{1}_F\}_{F\in\mathcal{F}}$ is in the vector-valued space $L^2(\ell^2; \sigma)$, and property (4) says that the control on averages is nondecreasing on the stopping tree $\mathcal{F}$. We emphasize that we are *not* assuming in this definition the stronger property that there is $C>1$ such that $\alpha_{\mathcal{F}}(F')>C\alpha_{\mathcal{F}}(F)$ whenever $F',F\in\mathcal{F}$ with $F'\subsetneqq F$. Instead, the properties (2) and (3) substitute for this lack. Of course the stronger property *does* hold for the familiar *Calderón-Zygmund* stopping data determined by the following requirements for $C>1$,

$$\begin{aligned}\mathbb{E}_{F'}^{\sigma}|f| &> C\mathbb{E}_{F}^{\sigma}|f| \text{ whenever } F',F\in\mathcal{F} \text{ with } F'\subsetneqq F,\\ \mathbb{E}_{I}^{\sigma}|f| &\leq C\mathbb{E}_{F}^{\sigma}|f| \text{ for } I\in\mathcal{C}_F,\end{aligned}$$

which are themselves sufficiently strong to automatically force properties (2) and (3) with $\alpha_{\mathcal{F}}(F)=\mathbb{E}_F^{\sigma}|f|$.

We have the following useful consequence of (2) and (3) that says the sequence $\{\alpha_{\mathcal{F}}(F)\mathbf{1}_F\}_{F\in\mathcal{F}}$ has a *quasiorthogonal* property relative to f with a constant C_0' depending only on C_0:

$$\left\|\sum_{F\in\mathcal{F}}\alpha_{\mathcal{F}}(F)\mathbf{1}_F\right\|_{L^2(\sigma)}^2\leq C_0'\|f\|_{L^2(\sigma)}^2. \tag{1.10}$$

Indeed, the Carleson condition (2) implies a geometric decay in levels of the tree $\mathcal{F}$, namely that there are positive constants C_1 and ε, depending on C_0, such that if $\mathfrak{C}_{\mathcal{F}}^{(n)}(F)$ denotes the set of n^{th} generation children of F in $\mathcal{F}$,

$$\sum_{F'\in\mathfrak{C}_{\mathcal{F}}^{(n)}(F):}|F'|_{\sigma}\leq\left(C_12^{-\varepsilon n}\right)^2|F|_{\sigma}, \qquad \text{for all } n\geq 0 \text{ and } F\in\mathcal{F}.$$

From this we obtain that

$$\begin{aligned}\sum_{n=0}^{\infty}\sum_{F'\in\mathfrak{C}_{\mathcal{F}}^{(n)}(F):}\alpha_{\mathcal{F}}(F')|F'|_{\sigma} &\leq \sum_{n=0}^{\infty}\sqrt{\sum_{F'\in\mathfrak{C}_{\mathcal{F}}^{(n)}(F)}\alpha_{\mathcal{F}}(F')^2|F'|_{\sigma}}\,C_12^{-\varepsilon n}\sqrt{|F|_{\sigma}}\\ &\leq C_1\sqrt{|F|_{\sigma}}\,C_{\varepsilon}\sqrt{\sum_{n=0}^{\infty}2^{-\varepsilon n}\sum_{F'\in\mathfrak{C}_{\mathcal{F}}^{(n)}(F)}\alpha_{\mathcal{F}}(F')^2|F'|_{\sigma}},\end{aligned}$$

and hence that

$$\begin{aligned}&\sum_{F\in\mathcal{F}}\alpha_{\mathcal{F}}(F)\left\{\sum_{n=0}^{\infty}\sum_{F'\in\mathfrak{C}_{\mathcal{F}}^{(n)}(F)}\alpha_{\mathcal{F}}(F')|F'|_{\sigma}\right\}\\ \lesssim\ &\sum_{F\in\mathcal{F}}\alpha_{\mathcal{F}}(F)\sqrt{|F|_{\sigma}}\sqrt{\sum_{n=0}^{\infty}2^{-\varepsilon n}\sum_{F'\in\mathfrak{C}_{\mathcal{F}}^{(n)}(F)}\alpha_{\mathcal{F}}(F')^2|F'|_{\sigma}}\\ \lesssim\ &\left(\sum_{F\in\mathcal{F}}\alpha_{\mathcal{F}}(F)^2|F|_{\sigma}\right)^{\frac12}\left(\sum_{n=0}^{\infty}2^{-\varepsilon n}\sum_{F\in\mathcal{F}}\sum_{F'\in\mathfrak{C}_{\mathcal{F}}^{(n)}(F)}\alpha_{\mathcal{F}}(F')^2|F'|_{\sigma}\right)^{\frac12}\\ \lesssim\ &\|f\|_{L^2(\sigma)}\left(\sum_{F'\in\mathcal{F}}\alpha_{\mathcal{F}}(F')^2|F'|_{\sigma}\right)^{\frac12}\lesssim\|f\|_{L^2(\sigma)}^2.\end{aligned}$$

This proves (1.10) since $\left\|\sum_{F\in\mathcal{F}}\alpha_{\mathcal{F}}(F)\mathbf{1}_F\right\|_{L^2(\sigma)}^2$ is dominated by twice the left hand side above.

Here is a basic reduction involving the NTV constant $\mathfrak{NTV}$.

Proposition 1.11. *Let*

$$\langle H_{\sigma}(f), g\rangle_{\omega} = \mathsf{H}_{near}(f,g) + \mathsf{H}_{disjoint}(f,g) + \mathsf{H}_{far}(f,g)$$

be a parallel corona decomposition as in (1.7) of the bilinear form $\langle H(f\sigma), g\rangle_{\omega}$ with stopping data $\mathcal{F}$ and $\mathcal{G}$ for f and g respectively. Then we have

$$\left|\langle H_{\sigma}(f), g\rangle_{\omega} - \mathsf{H}_{near}(f,g)\right| \lesssim (\mathfrak{NTV}) \|f\|_{L^2(\sigma)} \|g\|_{L^2(\omega)},$$

for all $f \in L^2(\sigma), g \in L^2(\omega)$.

We will use Proposition 1.11 in conjunction with a construction that permits *iteration* of general corona decompositions.

Lemma 1.12. *Suppose that $(C_0, \mathcal{F}, \alpha_{\mathcal{F}})$ constitutes* stopping data *for a function $f \in L^1_{loc}(\sigma)$, and that for each $F \in \mathcal{F}$, $\left(C_0, \mathcal{K}(F), \alpha_{\mathcal{K}(F)}\right)$ constitutes* stopping data *for the corona projection $\mathsf{P}^{\sigma}_{\mathcal{C}_F} f$. There is a positive constant C_1, depending only on C_0, such that if*

$$\begin{aligned} \mathcal{K}^*(F) &\equiv \left\{K \in \mathcal{K}(F) \cap \mathcal{C}_F : \alpha_{\mathcal{K}(F)}(K) \geq \alpha_{\mathcal{F}}(F)\right\} \\ \mathcal{K} &\equiv \bigcup_{F\in\mathcal{F}} \mathcal{K}^*(F) \cup \{F\}, \end{aligned}$$

$$\alpha_{\mathcal{K}}(K) \equiv \begin{array}{ccc} \alpha_{\mathcal{K}(F)}(K) & \text{for} & K \in \mathcal{K}^*(F) \setminus \{F\} \\ \max\left\{\alpha_{\mathcal{F}}(F), \alpha_{\mathcal{K}(F)}(F)\right\} & \text{for} & K = F \end{array}, \qquad \text{for } F \in \mathcal{F},$$

the triple $(C_1, \mathcal{K}, \alpha_{\mathcal{K}})$ constitutes stopping data *for f. We refer to the collection of intervals $\mathcal{K}$ as the* iterated *stopping times, and to the orthogonal decomposition $f = \sum_{K\in\mathcal{K}} P_{\mathcal{C}^{\mathcal{K}}_K} f$ as the* iterated *corona decomposition of f, where*

$$\mathcal{C}^{\mathcal{K}}_K \equiv \{I \in \mathcal{D} : I \subset K \text{ and } I \not\subset K' \text{ for } K' \prec_{\mathcal{K}} K\}.$$

Note that in our definition of $(C_1, \mathcal{K}, \alpha_{\mathcal{K}})$ we have 'discarded' from $\mathcal{K}(F)$ all of those $K \in \mathcal{K}(F)$ that are not in the corona $\mathcal{C}_F$, and also all of those $K \in \mathcal{K}(F)$ for which $\alpha_{\mathcal{K}(F)}(K)$ is strictly less than $\alpha_{\mathcal{F}}(F)$. Then the union of over F of what remains is our new collection of stopping times. We then define stopping data $\alpha_{\mathcal{K}}(K)$ according to whether or not $K \in \mathcal{F}$: if $K \notin \mathcal{F}$ but $K \in \mathcal{C}_F$ then $\alpha_{\mathcal{K}}(K)$ equals $\alpha_{\mathcal{K}(F)}(K)$, while if $K \in \mathcal{F}$, then $\alpha_{\mathcal{K}}(K)$ is the larger of $\alpha_{\mathcal{K}(F)}(F)$ and $\alpha_{\mathcal{F}}(K)$.

Proof. The monotonicity property (4) for the triple $(C_1, \mathcal{K}, \alpha_{\mathcal{K}})$ is obvious from the construction of $\mathcal{K}$ and $\alpha_{\mathcal{K}}(K)$. To establish property (1), we must distinguish between the various coronas $\mathcal{C}^{\mathcal{K}}_K$, $\mathcal{C}^{\mathcal{K}(F)}_K$ and $\mathcal{C}^{\mathcal{F}}_K$ that could be associated with $K \in \mathcal{K}$, when K belongs to any of the stopping trees $\mathcal{K}$, $\mathcal{K}(F)$ or $\mathcal{F}$. Suppose now that $I \in \mathcal{C}^{\mathcal{K}}_K$ for some $K \in \mathcal{K}$. Then there is a unique $F \in \mathcal{F}$ such that $\mathcal{C}^{\mathcal{K}}_K \subset \mathcal{C}^{\mathcal{K}(F)}_K \subset \mathcal{C}^{\mathcal{F}}_F$, and so $\mathbb{E}^{\sigma}_I |f| \leq \alpha_{\mathcal{F}}(F)$ by property (1) for the triple $(C_0, \mathcal{F}, \alpha_{\mathcal{F}})$. Then $\alpha_{\mathcal{F}}(F) \leq \alpha_{\mathcal{K}}(K)$ follows from the definition of $\alpha_{\mathcal{K}}(K)$, and we have property (1) for the triple $(C_1, \mathcal{K}, \alpha_{\mathcal{K}})$. Property (2) holds for the triple $(C_1, \mathcal{K}, \alpha_{\mathcal{K}})$ since if $K \in \mathcal{C}^{\mathcal{F}}_F$, then

$$\begin{aligned} \sum_{K' \preceq_{\mathcal{K}} K} |K'|_{\sigma} &= \sum_{K' \in \mathcal{K}(F):\ K' \subset K} |K'|_{\sigma} + \sum_{F' \prec_{\mathcal{F}} F:\ F' \subset K}\ \sum_{K' \in \mathcal{K}(F')} |K'|_{\sigma} \\ &\leq C_0^2 |K|_{\sigma} + \sum_{F' \prec_{\mathcal{F}} F:\ F' \subset K} C_0^2 |F'|_{\sigma} \leq 2C_0^2 |K|_{\sigma}. \end{aligned}$$

Finally, property (3) holds for the triple $(C_1, \mathcal{K}, \alpha_{\mathcal{K}})$ since

$$\begin{aligned}\sum_{K\in\mathcal{K}} \alpha_{\mathcal{K}}(K)^2 |K|_\sigma &= \sum_{F\in\mathcal{F}} \sum_{K\in\mathcal{K}(F)} \alpha_{\mathcal{K}(F)}(K)^2 |K|_\sigma \\ &\leq \sum_{F\in\mathcal{F}} C_0^2 \left\|\mathsf{P}^\sigma_{\mathcal{C}_F} f\right\|^2_{L^2(\sigma)} \leq C_0^2 \|f\|^2_{L^2(\sigma)} .\end{aligned}$$

□

2. Bounded fluctuation and functional energy

In the proof of Theorem 1 it will be convenient to isolate two intermediate notions that guide the philosophy of the proof, namely minimal bounded fluctuation functions, and the functional energy conditions.

2.1. **Bounded fluctuation.** The notion of bounded fluctuation is an extension of the notion of bounded function intermediate between L^∞ and BMO^{dyadic}. There are various versions of bounded fluctuation functions, and conditions defined in terms of them, that arise in the course of our investigation. We start with a definition of bounded fluctuation that is closely tied to the corona projections in the CZ corona decomposition.

Definition 2.1. *Given $\gamma > 0$, an interval $K \in \mathcal{D}^\sigma$, and a function f supported on K, we say that f is a function of* bounded fluctuation *on K, written $f \in \mathcal{BF}^{(\gamma)}_\sigma(K)$, if there is a pairwise disjoint collection $\mathcal{K}_f$ of $\mathcal{D}^\sigma$-subintervals of K such that*

$$\begin{aligned}\int_K f\sigma &= 0, \\ f &= a_{K'} \text{ (a constant) on } K', \qquad K' \in \mathcal{K}_f, \\ |a_{K'}| &> \gamma, \qquad K' \in \mathcal{K}_f, \\ \frac{1}{|I|_\sigma}\int_I |f|\,\sigma &\leq 1, \qquad I \in \widehat{\mathcal{K}_f},\end{aligned}$$

where

$$\widehat{\mathcal{K}_f} = \{I \in \mathcal{D}^\sigma : I \subset K \text{ and } I \not\subset K' \text{ for any } K' \in \mathcal{K}_f\}$$

is the corona determined by K and $\mathcal{K}_f$.

In the case $\gamma > 1$, we see that f is of bounded fluctuation on K if it is supported in K with mean zero, and equals a constant $a_{K'}$ of modulus greater than γ on any subinterval K' where $\mathbb{E}^\sigma_{K'}|f| > 1$. Thus in the case $\gamma > 1$, the collection of distinguished intervals is uniquely determined, but in general $\mathcal{K}_f$ must be specified. If we also require in Definition 2.1 that

$$a_{K'} > \gamma, \qquad K' \in \mathcal{K}_f,$$

then we denote the resulting collection of functions by $\mathcal{PBF}^{(\gamma)}_\sigma(K)$ and refer to such an f as a function of *positive* bounded fluctuation on K.

Now we observe that for $f \in \mathcal{BF}^{(\gamma)}_\sigma(K)$ with $\gamma > 1$, the Haar support *supp* $\widehat{f}$ of f contains the set $\pi\mathcal{K}_f$ of parents of the intervals in $\mathcal{K}_f$. Indeed, if $K' \in \mathcal{K}_f$, then the expected values of f on K' and its sibling $\theta K'$ necessarily differ, which implies that $\pi K'$ is in the Haar support of $\widehat{f}$. More precisely, on K' we have

$$\triangle^\sigma_{\pi K'} f = a_{K'} - \mathbb{E}^\sigma_{\pi K'} f,$$

and since $|\mathbb{E}^{\sigma}_{\pi K'} f| \leq \mathbb{E}^{\sigma}_{\pi K'} |f| \leq 1$, we have $|\triangle^{\sigma}_{\pi K'} f| \, |_{K'} \geq |a_{K'}| - 1 \geq \gamma - 1 > 0$. It turns out to be a crucial reduction in our proof of Theorem 1 that we can restrict attention to functions f of bounded fluctuation that have *minimal* Haar support $supp\ \widehat{f}$, namely equal to $\pi\mathcal{K}_f$. More precisely, define $\widehat{f} : \mathcal{D} \to \mathbb{C}$ by $\widehat{f}(I) \equiv \langle f, h^{\sigma}_I \rangle_{\sigma}$ to be the Haar coefficient map (with underlying measure σ being understood), and

$$\pi\mathcal{K}_f \equiv \{\pi_{\mathcal{D}} K' : K' \in \mathcal{K}_f\} .$$

It will however be necessary to relax the requirement that γ be large, and instead require $K' = (\pi_{\mathcal{D}} K')_{small}$ for $K' \in \mathcal{K}_f$. Here the two dyadic children of I are defined as I_{small} and I_{big} where $|I_{small}|_{\sigma} \leq |I_{big}|_{\sigma}$.

Definition 2.2. *For $\gamma > 0$, define the collection $\mathcal{MBF}^{(\gamma)}_{\sigma}(K)$ of functions of* minimal *bounded fluctuation by*

$$\left\{ f \in \mathcal{PBF}^{(\gamma)}_{\sigma}(K) : supp\ \widehat{f} \subset \pi\mathcal{K}_f \text{ and } K' = (\pi_{\mathcal{D}} K')_{small} \text{ for } K' \in \mathcal{K}_f \right\}.$$

Thus the functions $f \in \mathcal{MBF}_{\sigma}(K)$ have their Haar support *supp* $\widehat{f}$ as small as possible given that they satisfy the conditions for belonging to $\mathcal{PBF}_{\sigma}(K)$. Moreover, the distinguished intervals K' in $\mathcal{K}_f$ are the *small* child of their parent - a property that is a consequence of $a_{K'} > \gamma$ if $\gamma \geq 1$, but in general must be included in the definition. Note that while $\mathcal{K}_f$ consists of pairwise disjoint intervals for $f \in \mathcal{MBF}_{\sigma}(K)$, the collection of parents $\pi\mathcal{K}_f$ may have considerable overlap, and this represents the main difficulty in dealing with functions of minimal bounded fluctuation. We use the term *restricted bounded fluctuation on* K to designate a function f that is *either* bounded by 1 in modulus on K, *or* is of minimal bounded fluctuation on K; i.e.

$$f \in \mathcal{RBF}^{(\gamma)}_{\sigma}(K) \equiv (L^{\infty}_K)_1(\sigma) \bigcup \mathcal{MBF}^{(\gamma)}_{\sigma}(K) .$$

The final key element in our proof of Theorem 1 is an estimate for a highly nonlinear form $\mathsf{B}_{stop}(f,g)$ with either $f \in \mathcal{MBF}_{\sigma}$ or $g \in \mathcal{MBF}_{\omega}$, and a bound on the stopping energy $\boldsymbol{X}(f,g)$, that exploits the interval size splitting of NTV. See Lemma 6.4.

2.2. **Functional energy.** In the proof of Theorem 1 it will be convenient to isolate the following intermediate notion that guides the philosophy of the proof, namely the functional energy conditions.

Definition 2.3. *A collection $\mathcal{F}$ of dyadic intervals is σ-*Carleson *if*

$$\sum_{F \in \mathcal{F}:\ F \subset S} |F|_{\sigma} \leq C_{\mathcal{F}} |S|_{\sigma} , \qquad S \in \mathcal{F}.$$

The constant $C_{\mathcal{F}}$ is referred to as the Carleson norm of $\mathcal{F}$.

Definition 2.4. *Let $\mathcal{F}$ be a collection of dyadic intervals. A collection of functions $\{g_F\}_{F \in \mathcal{F}}$ in $L^2(\omega)$ is said to be $\mathcal{F}$-*adapted *if for each $F \in \mathcal{F}$ there is a collection $\mathcal{J}(F)$ of intervals in $\mathcal{D}^{\omega}$ such that*

$$\mathcal{J}(F) \subset \{J \in \mathcal{D}^{\omega} : J \Subset F\}$$

($J \Subset F$ implies J is (ϵ, r)-good with respect to $\mathcal{D}^{\sigma} = \mathcal{D}^{\omega}$ as in Subsection 3.3 below) and if $\mathcal{J}^*(F)$ consists of the maximal *intervals J in the collection $\mathcal{J}(F)$, then each of the following three conditions hold:*

(1) *for each* $F\in\mathcal{F}$, *the Haar coefficients* $\widehat{g_F}(J)=\langle g_F,h_J^{\omega}\rangle_{\omega}$ *of* g_F *are non-negative and supported in* $\mathcal{J}(F)$, *i.e.*

$$\left\{\begin{array}{lll}\widehat{g_F}(J)\geq 0 & \text{for all} & J\in\mathcal{J}(F)\\ \widehat{g_F}(J)=0 & \text{for all} & J\notin\mathcal{J}(F)\end{array}\right. ,\qquad F\in\mathcal{F},$$

(2) *the collection* $\{g_F\}_{F\in\mathcal{F}}$ *is pairwise orthogonal in* $L^2(\omega)$,

(3) *there is a positive constant* C *such that for every interval* I *in* $\mathcal{D}^{\sigma}$, *the collection of intervals*

$$\mathcal{B}_I\equiv\{J^*\subset I: J^*\in\mathcal{J}^*(F)\ \text{for some } F\supset I\}$$

has overlap bounded by C, *i.e.* $\sum_{J^*\in\mathcal{B}_I}\mathbf{1}_{J^*}\leq C$, *for all* $I\in\mathcal{D}^{\sigma}$.

Note that condition (2) holds if the collections $\mathcal{J}(F)$ are pairwise disjoint for $F\in\mathcal{F}$.

Definition 2.5. *Let* $\mathfrak{F}$ *be the smallest constant in the '**f**unctional energy' inequality below, holding for all non-negative* $h\in L^2(\sigma)$, *all* σ*-Carleson collections* $\mathcal{F}$, *and all* $\mathcal{F}$*-adapted collections* $\{g_F\}_{F\in\mathcal{F}}$, *and where* $\mathcal{J}^*(F)$ *consists of the* maximal *intervals* J *in the collection* $\mathcal{J}(F)$:

(2.6)
$$\sum_{F\in\mathcal{F}}\sum_{J^*\in\mathcal{J}^*(F)}\mathsf{P}(J^*,h\sigma)\left|\left\langle\frac{x}{|J^*|},g_F\mathbf{1}_{J^*}\right\rangle_{\omega}\right|\leq\mathfrak{F}\|h\|_{L^2(\sigma)}\left[\sum_{F\in\mathcal{F}}\|g_F\|_{L^2(\omega)}^2\right]^{1/2}.$$

We refer to this as the *functional energy condition*. There is of course a dual version of this condition as well with constant $\mathfrak{F}^*$.

2.3. **Outline of the proof.** The disjoint form $\mathsf{H}_{disjoint}(f,g)$ in (1.7) is easily controlled by the strong $\mathcal{A}_2$ condition and the interval testing conditions using Lemma 3.1 in Section 3:

$$|\mathsf{H}_{disjoint}(f,g)|\lesssim(\mathfrak{NTV})\|f\|_{L^2(\sigma)}\|g\|_{L^2(\omega)}.$$

We show in Section 5 that after further corona decompositions, the near and far forms satisfy

$$\begin{array}{rcl}|\mathsf{H}_{near}(f,g)| & \lesssim & (\mathfrak{NTV}_{ind}+\mathfrak{M}+\mathfrak{M}^*)\|f\|_{L^2(\sigma)}\|g\|_{L^2(\omega)},\\ |\mathsf{H}_{far}(f,g)| & \lesssim & (\mathfrak{NTV})\|f\|_{L^2(\sigma)}\|g\|_{L^2(\omega)},\end{array}$$

where $\mathfrak{M}$ and $\mathfrak{M}^*$ are the best constants in a bilinear minimal bounded fluctuation inequality (5.39) and its dual, and with the bulk of the work in estimating the far form H_{far} taken up in proving the Interwining Proposition in Section 4.

Finally, in Section 6 we use the Intertwining Proposition 4.2 to reduce the *bi*linear minimal bounded fluctuation condition (5.39) to a similar inequality (6.5), but for a *non*linear form B_{stop} that is esssentially the stopping term introduced by NTV in [NTV4], i.e.

$$\mathfrak{M}+\mathfrak{M}^*\lesssim\mathfrak{NTV}+\mathfrak{B}_{stop}^{minimal}+\mathfrak{B}_{stop}^{minimal\ *},$$

where $\mathfrak{B}_{stop}^{minimal}$ is the best constant in (6.5). Finally, the full force of the special structure of minimal bounded fluctuation functions is exploited along with energy control, to obtain

$$\mathfrak{B}_{stop}^{minimal}+\mathfrak{B}_{stop}^{minimal\ *}\lesssim\mathfrak{NTV},$$

and this completes the proof of Theorem 1. Thus the only place where indicator/interval testing is used over interval testing, is in reducing control of the near form H_{near} to the bilinear minimal bounded fluctutation conditions (5.39).

3. Preliminaries of the proof

A crucial reduction of Problem 1.1 is delivered by the following lemma due to Nazarov, Treil and Volberg (see [NTV4] and [Vol]).

Lemma 3.1. *For $f \in L^2(\sigma)$ and $g \in L^2(\omega)$ we have*

$$\begin{aligned}\sum_{\substack{(I,J)\in\mathcal{D}^\sigma\times\mathcal{D}^\omega\\ 2^{-r}|I|\leq|J|\leq 2^r|I|}} \left|\left\langle H_\sigma\left(\triangle_I^\sigma f\right),\triangle_J^\omega g\right\rangle_\omega\right| &\lesssim (\mathfrak{NTV})\left\|f\right\|_{L^2(\sigma)}\left\|g\right\|_{L^2(\omega)}\\ \sum_{\substack{(I,J)\in\mathcal{D}^\sigma\times\mathcal{D}^\omega\\ I\cap J=\emptyset \text{ and } \frac{|J|}{|I|}\notin\left[2^{-r},2^r\right]}} \left|\left\langle H_\sigma\left(\triangle_I^\sigma f\right),\triangle_J^\omega g\right\rangle_\omega\right| &\lesssim \sqrt{\mathcal{A}_2}\left\|f\right\|_{L^2(\sigma)}\left\|g\right\|_{L^2(\omega)}.\end{aligned}$$

Proof. To prove the first assertion we split the sum into two pieces,

$$\left\{\sum_{\substack{2^{-r}|I|\leq|J|\leq 2^r|I|\\ dist(J,I)\leq 2^{r+1}|I|}} + \sum_{\substack{2^{-r}|I|\leq|J|\leq 2^r|I|\\ dist(J,I)>2^{r+1}|I|}}\right\}\left|\left\langle H_\sigma\left(\triangle_I^\sigma f\right),\triangle_J^\omega g\right\rangle_\omega\right|.$$

The first sum here is handled using the argument for the diagonal short-range terms in Subsection 9.2 of [LaSaUr], and the second sum here is handled by the argument for the long-range terms in Subsection 9.4 of [LaSaUr].

Now we turn to the second assertion. By duality it suffices to consider only $|J| \leq 2^{-r}|I|$ in the sum on the left of the second assertion, and we split the resulting sum into two pieces:

$$\left\{\sum_{J\subset 3I\setminus I \text{ and } |J|\leq 2^{-r}|I|} + \sum_{J\cap 3I=\emptyset \text{ and } |J|\leq 2^{-r}|I|}\right\}\left|\left\langle H_\sigma\left(\triangle_I^\sigma f\right),\triangle_J^\omega g\right\rangle_\omega\right|.$$

These sums are estimated using the arguments for the mid-range and long-range terms in Subsections 9.3 and 9.4 respectively of [LaSaUr]. ☐

3.1. **Monotonicity property.**

The following Monotonicity Property for the Hilbert transform will also play an important role in proving our theorem.

Lemma 3.2 (Monotonicity Property)**.** *Suppose that ν is a signed measure, and μ is a positive measure with $\mu \geq |\nu|$, both supported outside an interval J. Suppose also that φ is a function supported in J with $\int \varphi d\omega = 0$ and such that there is $c \in J$ such that*

$$\begin{aligned}\varphi(y) &\leq 0 \text{ for } y\leq c,\\ \varphi(y) &\geq 0 \text{ for } y\geq c.\end{aligned}$$

Then

$$\left|\left\langle H\nu,\varphi\right\rangle_\omega\right| \leq \left\langle H\mu,\varphi\right\rangle_\omega.$$

Now let $J \subset J^* \subset 2J^* \subset I$, and suppose in addition that $\varphi = h_J^\omega$ and that μ is supported outside I. Then we have the pointwise estimate

$$\begin{aligned}\mathbf{E} &\equiv \langle H\mu, h_J^\omega \rangle_\omega - \frac{1}{|J^*|} \mathsf{P}(J^*, \mu) \ \langle x, h_J^\omega \rangle_\omega \\ &= O\left(\frac{|J|}{|J^*|^2} \widetilde{\mathsf{P}}(J^*, \mu) \ \langle x, h_J^\omega \rangle_\omega \right),\end{aligned}$$

where

$$\begin{aligned}\widetilde{\mathsf{P}}(K, \mu) &\equiv \int_{\mathbb{R}} \frac{|K|^2}{(|K| + |y - c(K)|)^3} d\mu(y), \\ \langle x, h_J^\omega \rangle_\omega &= \langle x - c(J), h_J^\omega \rangle_\omega = \int_J (x - c(J)) h_J^\omega(x) d\omega \ge 0.\end{aligned} \tag{3.3}$$

Moreover, there is $\gamma > 2$ such that if in addition $\gamma J^* \subset I$, then

$$\langle H\mu, h_J^\omega \rangle_\omega \approx \frac{1}{|J^*|} \mathsf{P}(J^*, \mu) \ \langle x, h_J^\omega \rangle_\omega . \tag{3.4}$$

Remark 3.5. *This monotonicity property will be applied when $\varphi = h_J^\omega$ is a Haar function adapted to J, in which case the point c can be taken to be the center of J.*

Proof. Let $J_- = J \cap (-\infty, c)$ and $J_+ = J \cap (c, \infty)$. We may assume that

$$\int_{J_-} |\varphi| \, d\omega = \int_{J_+} |\varphi| \, d\omega = 1.$$

Then we have

$$\begin{aligned}\langle H\nu, \varphi \rangle_\omega &= \int_{J_+} H\nu(x) \varphi(x) d\omega(x) + \int_{J_-} H\nu(x) \varphi(x) d\omega(x) \\ &= \int_{J_+} H\nu(x) |\varphi(x)| d\omega(x) - \int_{J_-} H\nu(x') |\varphi(x')| d\omega(x') \\ &= \int_{J_+} \int_{J_-} [H\nu(x) - H\nu(x')] |\varphi(x')| d\omega(x') |\varphi(x)| d\omega(x) \\ &= \int_{J_+} \int_{J_-} \int_{\mathbb{R} \setminus J} \frac{x - x'}{(y - x)(y - x')} d\nu(y) |\varphi(x')| d\omega(x') |\varphi(x)| d\omega(x),\end{aligned}$$

and since $\frac{x-x'}{(y-x)(y-x')} \ge 0$ for $y \in \mathbb{R} \setminus J$ and $x \in J_+$ and $x' \in J_-$, we have

$$\begin{aligned}& |\langle H\nu, \varphi \rangle_\omega| \\ \le & \int_{J_+} \int_{J_-} \int_{\mathbb{R} \setminus J} \frac{x - x'}{(y - x)(y - x')} d\mu(y) |\varphi(x')| d\omega(x') |\varphi(x)| d\omega(x) \\ = & \langle H\mu, \varphi \rangle_\omega ,\end{aligned}$$

where the last equality follows from the previous display with μ in place of ν.

Now suppose that $J\subset J^{\ast}\subset 2J^{\ast}\subset I$, $\varphi=h_{J}^{\omega}$ and μ,ν are supported outside I. Then for $x\in J_{+}$, $x^{\prime}\in J_{-}$ and $y\notin I$, we have

$$\begin{aligned}&\frac{x-x^{\prime}}{\left(y-x\right)\left(y-x^{\prime}\right)}-\frac{x-x^{\prime}}{\left|J^{\ast}\right|}\frac{\left|J^{\ast}\right|}{\left|y-c\left(J^{\ast}\right)\right|^{2}}\\ =\;&\left(x-x^{\prime}\right)\frac{\left(y-c\left(J^{\ast}\right)\right)^{2}-\left(y-x\right)\left(y-x^{\prime}\right)}{\left(y-x\right)\left(y-x^{\prime}\right)\left(y-c\left(J^{\ast}\right)\right)^{2}}\\ =\;&O\left(\left|x-x^{\prime}\right|\frac{\left|J^{\ast}\right|}{\left|y-c\left(J^{\ast}\right)\right|^{3}}\right).\end{aligned}$$

Now we recall that

$$A\equiv\int_{J_{-}}\left|h_{J}^{\omega}\right|d\omega=\int_{J_{+}}\left|h_{J}^{\omega}\right|d\omega=\sqrt{\frac{\left|J_{-}\right|_{\omega}\left|J_{+}\right|_{\omega}}{\left|J\right|_{\omega}}},$$

so that with $\varphi=\frac{1}{A}h_{J}^{\omega}$, we obtain

$$\begin{aligned}&\left\langle H\mu,h_{J}^{\omega}\right\rangle_{\omega}=A\ \left\langle H\mu,\varphi\right\rangle_{\omega}\\ =\;&\frac{1}{A}\int_{J_{+}}\int_{J_{-}}\int_{\mathbb{R}\setminus J}\frac{x-x^{\prime}}{\left(y-x\right)\left(y-x^{\prime}\right)}d\mu\left(y\right)\left|h_{J}^{\omega}\left(x^{\prime}\right)\right|d\omega\left(x^{\prime}\right)\left|h_{J}^{\omega}\left(x\right)\right|d\omega\left(x\right)\\ =\;&\frac{1}{A}\int_{J_{+}}\int_{J_{-}}\int_{\mathbb{R}\setminus J}\frac{x-x^{\prime}}{\left|J^{\ast}\right|}\frac{\left|J^{\ast}\right|}{\left|y-c\left(J^{\ast}\right)\right|^{2}}d\mu\left(y\right)\left|h_{J}^{\omega}\left(x^{\prime}\right)\right|d\omega\left(x^{\prime}\right)\left|h_{J}^{\omega}\left(x\right)\right|d\omega\left(x\right)\\ &+O\left(\int_{J_{+}}\int_{J_{-}}\int_{\mathbb{R}\setminus J}\left|x-x^{\prime}\right|\frac{\left|J^{\ast}\right|}{\left|y-c\left(J^{\ast}\right)\right|^{3}}d\mu\left(y\right)\left|h_{J}^{\omega}\left(x^{\prime}\right)\right|d\omega\left(x^{\prime}\right)\left|h_{J}^{\omega}\left(x\right)\right|d\omega\left(x\right)\right)\\ =\;&\frac{1}{\left|J^{\ast}\right|}\mathsf{P}\left(J^{\ast},\mu\right)\ \left\langle x,h_{J}^{\omega}\right\rangle_{\omega}+O\left(\frac{\left|J\right|}{\left|J^{\ast}\right|^{2}}\widetilde{\mathsf{P}}\left(J^{\ast},\mu\right)\ \left\langle x,h_{J}^{\omega}\right\rangle_{\omega}\right),\end{aligned}$$

since $\left|h_{J}^{\omega}\left(x^{\prime}\right)\right|\left|h_{J}^{\omega}\left(x\right)\right|=-h_{J}^{\omega}\left(x^{\prime}\right)h_{J}^{\omega}\left(x\right)$,

$$\begin{aligned}&\frac{1}{A}\int_{J_{+}}\int_{J_{-}}\int_{\mathbb{R}\setminus J}\frac{x}{\left|J^{\ast}\right|}\frac{\left|J^{\ast}\right|}{\left|y-c\left(J^{\ast}\right)\right|^{2}}d\mu\left(y\right)h_{J}^{\omega}\left(x^{\prime}\right)d\omega\left(x^{\prime}\right)h_{J}^{\omega}\left(x\right)d\omega\left(x\right)\\ =\;&-\frac{1}{\left|J^{\ast}\right|}\mathsf{P}\left(J^{\ast},\mu\right)\int_{J_{+}}xh_{J}^{\omega}\left(x\right)d\omega\left(x\right),\end{aligned}$$

$$\begin{aligned}&-\frac{1}{A}\int_{J_{+}}\int_{J_{-}}\int_{\mathbb{R}\setminus J}\frac{x^{\prime}}{\left|J^{\ast}\right|}\frac{\left|J^{\ast}\right|}{\left|y-c\left(J^{\ast}\right)\right|^{2}}d\mu\left(y\right)\left|h_{J}^{\omega}\left(x^{\prime}\right)\right|d\omega\left(x^{\prime}\right)\left|h_{J}^{\omega}\left(x\right)\right|d\omega\left(x\right)\\ =\;&\frac{1}{\left|J^{\ast}\right|}\mathsf{P}\left(J^{\ast},\mu\right)\int_{J_{-}}x^{\prime}h_{J}^{\omega}\left(x^{\prime}\right)d\omega\left(x^{\prime}\right),\end{aligned}$$

and

$$\int_{J_{+}}xh_{J}^{\omega}\left(x\right)d\omega\left(x\right)+\int_{J_{-}}x^{\prime}h_{J}^{\omega}\left(x^{\prime}\right)d\omega\left(x^{\prime}\right)=\left\langle x,h_{J}^{\omega}\right\rangle_{\omega}.$$

Finally, (3.4) follows easily from the above pointwise estimate for γ large enough.

□

3.2. **Energy lemma.** We formulate a refinement of the Energy Lemma from [LaSaUr]. First recall that the energy $\mathsf{E}(J,\omega)$ of ω on the interval J is given by

$$\mathsf{E}(J,\omega)^2 \equiv \frac{1}{|J|_\omega}\int_J \left(\mathbb{E}_J^{\omega(dx')}\frac{x-x'}{|J|}\right)^2 d\omega(x),$$

and the corresponding functional $\Phi(J,\nu)$ by

$$\Phi(J,\nu) \equiv \omega(J)\,\mathsf{E}(J,\omega)^2\,\mathsf{P}(J,|\nu|)^2,$$

where ν is a signed measure on $\mathbb{R}$. The following *Energy Condition* was proved in [LaSaUr]: for all intervals I,

$$\sum_{n=1}^{\infty}\Phi(J_n,\nu) \lesssim \left(\sqrt{\mathcal{A}_2}+\mathfrak{T}\right)|I|_\sigma, \qquad \dot{\bigcup}_{n=1}^{\infty} J_n \subset I. \tag{3.6}$$

Suppose now we are given an interval $J \in \mathcal{D}^\omega$, and a subset $\mathcal{H}$ of the dyadic subgrid $\mathcal{D}^\omega(J)$ of intervals from $\mathcal{D}^\omega$ that are contained in J. Let $\mathsf{P}_{\mathcal{H}}^\omega = \sum_{J'\in\mathcal{H}} \triangle_{J'}^\omega$ be the ω-Haar projection onto $\mathcal{H}$ and define the $\mathcal{H}$-energy $\mathsf{E}_{\mathcal{H}}(J,\omega)$ of ω on the interval J by

$$\begin{aligned}
\mathsf{E}_{\mathcal{H}}(J,\omega)^2 &\equiv \frac{1}{|J|_\omega}\int_J \left(\mathbb{E}_J^{\omega(dx')}\frac{\mathsf{P}_{\mathcal{H}}^{\omega(dx)}(x-x')}{|J|}\right)^2 d\omega(x) \\
&= \frac{1}{|J|_\omega}\int_J \left(\frac{\mathsf{P}_{\mathcal{H}}^\omega x}{|J|}\right)^2 d\omega(x) \\
&= \frac{1}{|J|_\omega}\sum_{J'\in\mathcal{H}}\left|\left\langle \frac{x}{|J|}, h_{J'}^\omega\right\rangle_\omega\right|^2 = \frac{1}{|J|_\omega |J|^2}\sum_{J'\in\mathcal{H}}\left|\widehat{x}(J')\right|^2 .
\end{aligned}\tag{3.7}$$

For ν a signed measure on $\mathbb{R}$, and $\mathcal{H}$ a subset of the dyadic grid $\mathcal{D}^\omega$, we define the functional

$$\Phi_{\mathcal{H}}(J,\nu) \equiv \omega(J)\,\mathsf{E}_{\mathcal{H}}(J,\omega)^2\,\mathsf{P}(J,|\nu|)^2 .$$

We need yet another property peculiar to the Hilbert transform kernel

$$K(x,y) = \frac{1}{x-y} = K^x(y) = K_y(x).$$

Lemma 3.8. *Suppose J is an interval with center c_J, choose $y \notin J$, and let $\eta = \frac{|y-c_J|}{|J|/2} > 1$. Then*

$$\left|\langle K_y, h_J^\omega\rangle_\omega + \frac{\langle x, h_J^\omega\rangle_\omega}{|y-c_J|^2}\right| \le \frac{1}{\eta-1}\frac{\langle x, h_J^\omega\rangle_\omega}{|y-c_J|^2}.$$

Proof. With c_J the center of J, we have using $\int h_J^\omega(x)\,d\omega(x) = 0$ that

$$\begin{aligned}
\langle K_y, h_J^\omega\rangle_\omega &= \int \frac{1}{x-y}h_J^\omega(x)\,d\omega(x) = \int \frac{1}{(x-c_J)-(y-c_J)}h_J^\omega(x)\,d\omega(x) \\
&= -\frac{1}{y-c_J}\int \frac{1}{1-\frac{x-c_J}{y-c_J}}h_J^\omega(x)\,d\omega(x) \\
&= -\sum_{n=1}^{\infty}\int \frac{(x-c_J)^n}{(y-c_J)^{n+1}}h_J^\omega(x)\,d\omega(x).
\end{aligned}$$

Since the Haar function takes opposite signs on its two children we have without loss of generality,

$$\widehat{x}(J)=\int(x-c_J)\,h_J^{\omega}(x)\,d\omega(x)=\int|x-c_J|\,|h_J^{\omega}(x)|\,d\omega(x)\geq 0,$$

and for $n\geq 1$ we then have the estimate

$$\left|\int(x-c_J)^n\,h_J^{\omega}(x)\,d\omega(x)\right|\leq(|J|/2)^{n-1}\int|x-c_J|\,|h_J^{\omega}(x)|\,d\omega(x)=(|J|/2)^{n-1}\,\widehat{x}(J).$$

Consequently,

$$\left|\langle K_y,h_J^{\omega}\rangle_{\omega}+\frac{\langle x,h_J^{\omega}\rangle_{\omega}}{|y-c_J|^2}\right|\leq\frac{1}{|y-c_J|^2}\sum_{n=2}^{\infty}\left(\frac{|J|/2}{y-c_J}\right)^{n-1}\widehat{x}(J)=\frac{1}{(y-c_J)^2}\frac{1}{\eta-1}\widehat{x}(J).$$

□

Thus in the Taylor expansion for the inner product $\langle K_y,h_J^{\omega}\rangle_{\omega}$, the linear term dominates.

Lemma 3.9 (**Energy Lemma**). *Let J be an interval in $\mathcal{D}^{\omega}$. Let Ψ_J be an $L^2(\omega)$ function supported in J and with ω-integral zero. Let ν be a signed measure supported in $\mathbb{R}\setminus 2J$ and denote the Haar support of Ψ_J by $\mathcal{H}=supp\widehat{\Psi_J}$. Then we have*

$$|\langle H(\nu),\Psi_J\rangle_{\omega}|\leq C\,\|\Psi_J\|_{L^2(\omega)}\,\Phi_{\mathcal{H}}(J,\nu)^{\frac{1}{2}}. \tag{3.10}$$

The L^2 formulation $|\langle H(\nu),\Psi_J\rangle_{\omega}|\leq C\,\|\Psi_J\|_{L^2(\omega)}\,\Phi(J,\nu)^{\frac{1}{2}}$ proves useful in many estimates. However, we will often apply this in its dual formulation. Namely, we have

$$\|H(\nu)-\mathbb{E}_J^{\omega}H(\nu)\|_{L^2(J,\omega)}\lesssim\Phi(J,\nu)^{\frac{1}{2}}. \tag{3.11}$$

Note that on the left, we are subtracting off the mean value, and only testing the $L^2(\omega)$ norm on J.

Proof. We calculate using Lemma 3.8 that

$$\begin{aligned}
|\langle H(\nu),\Psi_J\rangle_{\omega}| &= \left|\int_J\int_{\mathbb{R}\setminus 2J}\frac{1}{x-y}\;\Psi_J(x)\;d\nu(y)\;d\omega(x)\right|\\
&= \left|\int_J\int_{\mathbb{R}\setminus 2J}\frac{1}{x-y}\;\sum_{J'\in\mathcal{H}}\langle\Psi_J,h_{J'}^{\omega}\rangle_{\omega}\,h_{J'}^{\omega}(x)\;d\nu(y)\;d\omega(x)\right|\\
&= \left|\sum_{J'\in\mathcal{H}}\int_{\mathbb{R}\setminus 2J}\langle K_y,h_{J'}^{\omega}\rangle_{\omega}\;\;\widehat{\Psi_J}(J')\;\;d\nu(y)\right|\\
&\leq \sum_{J'\in\mathcal{H}}\int_{\mathbb{R}\setminus 2J}\frac{1}{\eta-1}\,\langle x,h_{J'}^{\omega}\rangle_{\omega}\,\frac{1}{|y-c_{J'}|^2}\;\left|\widehat{\Psi_J}(J')\right|\;d|\nu|(y).
\end{aligned}$$

Now use the approximation $\frac{1}{|y-c_{J'}|^2} \approx \frac{1}{|y-c_J|^2}$ for $J' \subset J$ and $y \in \mathbb{R} \setminus 2J$, so that the sum in J' becomes isolated. Then an application of Cauchy-Schwarz in J' yields

$$\begin{aligned} |\langle H(\nu), \Psi_J \rangle_\omega| &\lesssim \frac{1}{\eta - 1} \|\Psi_J\|_{L^2(\omega)} \left(\sum_{J' \in \mathcal{H}} |\langle x, h_{J'}^\omega \rangle_\omega|^2 \right)^{\frac{1}{2}} \int_{\mathbb{R} \setminus 2J} \frac{1}{|y-c_J|^2} d|\nu|(y) \\ &= \frac{1}{\eta - 1} \|\Psi_J\|_{L^2(\omega)} \left(\sum_{J' \in \mathcal{H}} |\widehat{x}(J')|^2 \right)^{\frac{1}{2}} \frac{1}{|J|} \mathsf{P}(J, |\nu|) \\ &= \frac{1}{\eta - 1} \|\Psi_J\|_{L^2(\omega)} \Phi_{\mathcal{H}}(J, \nu)^{\frac{1}{2}} . \end{aligned}$$

This completes the proof. □

Remark 3.12. *In the special case $\Psi_J = h_J^\omega$, we have $\mathcal{H} = \{J\}$ and (3.10) then gives*

$$|\langle H\nu, h_J^\omega \rangle_\omega| \lesssim \Phi_{\mathcal{H}}(J, \nu)^{\frac{1}{2}} = \left\langle \frac{x}{|J|}, h_J^\omega \right\rangle_\omega \mathsf{P}(J, |\nu|) . \tag{3.13}$$

We also need the following elementary Poisson estimate from [Vol] (see also [LaSaUr],'1 which is corrected here).

Lemma 3.14. *Suppose that $J \subset I \subset \widehat{I}$ and that* $\operatorname{dist}(J, \partial I) > \frac{1}{2}|J|^\varepsilon |I|^{1-\varepsilon}$. *Then*

$$|J|^{4\varepsilon-2} \mathsf{P}(J, \sigma \mathbf{1}_{\widehat{I} \setminus I})^2 \lesssim |I|^{4\varepsilon-2} \mathsf{P}(I, \sigma \mathbf{1}_{\widehat{I} \setminus I})^2. \tag{3.15}$$

Proof. We have

$$\mathsf{P}\left(J, \sigma \chi_{\widehat{I} \setminus I}\right) \approx \sum_{k=0}^{\infty} 2^{-k} \frac{1}{|2^k J|} \int_{(2^k J) \cap (\widehat{I} \setminus I)} d\sigma,$$

and $\left(2^k J\right) \cap \left(\widehat{I} \setminus I\right) \neq \emptyset$ requires

$$dist\,(J, e(I)) \leq \left|2^k J\right| .$$

Let k_0 be the smallest such k. By our distance assumption we must then have

$$|J|^\varepsilon |I|^{1-\varepsilon} \leq dist\,(J, e(I)) \leq 2^{k_0} |J| ,$$

or

$$2^{-k_0} \leq \left(\frac{|J|}{|I|} \right)^{1-\varepsilon} .$$

Now let k_1 be defined by $2^{k_1} \equiv \frac{|I|}{|J|}$. Then assuming $k_1 > k_0$ (the case $k_1 \leq k_0$ is similar) we have

$$\begin{aligned} \mathsf{P}\left(J, \sigma \chi_{\widehat{I} \setminus I}\right) &\approx \left\{ \sum_{k=k_0}^{k_1} + \sum_{k=k_1}^{\infty} \right\} 2^{-k} \frac{1}{|2^k J|} \int_{(2^k J) \cap (\widehat{I} \setminus I)} d\sigma \\ &\lesssim 2^{-k_0} \frac{|I|}{|2^{k_0} J|} \left(\frac{1}{|I|} \int_{(2^{k_1} J) \cap (\widehat{I} \setminus I)} d\sigma \right) + 2^{-k_1} \mathsf{P}\left(I, \sigma \chi_{\widehat{I} \setminus I}\right) \\ &\lesssim \left(\frac{|J|}{|I|} \right)^{1-2\varepsilon} \mathsf{P}\left(I, \sigma \chi_{\widehat{I} \setminus I}\right) + \frac{|J|}{|I|} \mathsf{P}\left(I, \sigma \chi_{\widehat{I} \setminus I}\right), \end{aligned}$$

which is the inequality (3.15). □

Finally, we need the following variant of Lemma 3.1 where we replace the Haar function in the inner product $\langle H_\sigma(\triangle_I^\sigma f), \triangle_J^\omega g\rangle$ (appearing in the statement of Lemma 3.1) corresponding to the larger interval by a bounded function dominated by an expectation, provided the larger interval is a stopping interval.

Lemma 3.16. *Suppose that all of the interval pairs $(I,J) \in \mathcal{D}^\sigma \times \mathcal{D}^\omega$ considered below are good. Suppose that $f \in L^2(\sigma)$ and $g \in L^2(\omega)$ and that $\mathcal{F}$ and $\mathcal{G}$ are σ-Carleson and ω-Carleson collections respectively as in Definition 2.3. Furthermore, suppose that for each pair of intervals $I \in \mathcal{D}^\sigma$ and $J \in \mathcal{D}^\omega$, there are bounded functions $\beta_{I,J}$ and $\gamma_{I,J}$ supported in $I \setminus 2J$ and $J \setminus 2I$ respectively, satisfying*

$$\left\|\beta_{I,J}\right\|_\infty, \left\|\gamma_{I,J}\right\|_\infty \leq 1.$$

Then we have

$$\begin{aligned}(3.17)\qquad & \sum_{\substack{(I,J)\in\mathcal{F}\times\mathcal{D}^\omega\\ I\cap J=\emptyset \text{ and } |J|\leq 2^{-r}|I|}} \left|\left\langle H_\sigma\left(\beta_{I,J}\mathbf{1}_I\mathbb{E}_I^\sigma f\right), \triangle_J^\omega g\right\rangle_\omega\right| \\ & + \sum_{\substack{(I,J)\in\mathcal{D}^\sigma\times\mathcal{G}\\ I\cap J=\emptyset \text{ and } |I|\leq 2^{-r}|J|}} \left|\left\langle H_\sigma\left(\triangle_I^\sigma f\right), \gamma_{I,J}\mathbf{1}_J\mathbb{E}_J^\omega g\right\rangle_\omega\right| \\ \lesssim\ & \sqrt{\mathcal{A}_2}\left\|f\right\|_{L^2(\sigma)}\left\|g\right\|_{L^2(\omega)},\end{aligned}$$

and also

$$\begin{aligned}&\sum_{\substack{(I,J)\in\mathcal{F}\times\mathcal{D}^\omega\\ 2^{-r}|I|\leq|J|\leq|I|}} \left|\left\langle H_\sigma\left(\mathbf{1}_I\mathbb{E}_I^\sigma f\right), \triangle_J^\omega g\right\rangle_\omega\right| + \sum_{\substack{(I,J)\in\mathcal{D}^\sigma\times\mathcal{G}\\ 2^{-r}|J|\leq|I|\leq|J|}} \left|\left\langle H_\sigma\left(\triangle_I^\sigma f\right), \mathbf{1}_J\mathbb{E}_J^\omega g\right\rangle_\omega\right| \\ &\qquad\lesssim \mathfrak{NTV}\left\|f\right\|_{L^2(\sigma)}\left\|g\right\|_{L^2(\omega)}.\end{aligned}$$

Proof. (of Lemma 3.16) The proof of Lemma 3.16 follows the lines of the proof of Lemma 3.1, but using the 'almost orthogonality' property $\sum_{F\in\mathcal{F}}|F|_\sigma\left(\mathbb{E}_I^\sigma|f|\right)^2 \lesssim \|f\|_{L^2(\sigma)}^2$ in place of the orthonormality of the Haar system $\sum_{F\in\mathcal{F}}\left|\langle f, h_I^\sigma\rangle_\sigma\right|^2 = \|f\|_{L^2(\sigma)}^2$ - and similarly for $\|g\|_{L^2(\omega)}^2$.We prove only the case $(I,J) \in \mathcal{F}\times\mathcal{D}^\omega$ and $|I| \geq 2^r|J|$.

We split the first sum in (3.17) into two sums, namely a long-range sum where in addition $J \cap 3I = \emptyset$, and a mid-range sum where in addition $J \subset 3I \setminus I$. We begin with the proof for the long-range sum, namely we prove

$$\begin{aligned}(3.18)\qquad A_{long-range} &\equiv \sum_{\substack{(I,J)\in\mathcal{F}\times\mathcal{D}^\omega\\ 3I\cap J=\emptyset \text{ and } |J|\leq 2^{-r}|I|}} \left|\left\langle H_\sigma\left(\beta_{I,J}\mathbf{1}_I\mathbb{E}_I^\sigma f\right), \triangle_J^\omega g\right\rangle_\omega\right| \\ &\lesssim \sqrt{\mathcal{A}_2}\left\|f\right\|_{L^2(\sigma)}\left\|g\right\|_{L^2(\omega)}.\end{aligned}$$

We apply the Energy Lemma 3.9 to estimate the inner product $\left\langle H_\sigma(\beta_{I,J}\mathbf{1}_I), h_J^\omega\right\rangle_\omega$ using $\nu = \beta_{I,J}\mathbf{1}_I\sigma$ and $2J \cap supp\left(\beta_{I,J}\right) = \emptyset$. Since $|\nu| \leq \left\|\beta_{I,J}\right\|_\infty \mathbf{1}_I\sigma \leq \mathbf{1}_I\sigma$, the Energy Lemma applies to give us the estimate below.

$$\begin{aligned}\beta(I,J) &\equiv \left|\left\langle H_\sigma(\beta_{I,J}\mathbf{1}_I), h_J^\omega\right\rangle_\omega\right| \lesssim \sqrt{|J|_\omega}\mathsf{P}(J,|\nu|) \\ &\lesssim \sqrt{|J|_\omega}\mathsf{P}(J,\mathbf{1}_I\sigma) \lesssim \sqrt{|J|_\omega}\,|I|_\sigma\cdot\frac{|J|}{dist(I,J)^2}.\end{aligned}$$

We have used the inequality $\mathsf{P}(J,\mathbf{1}_I\sigma)\lesssim\frac{|J|}{dist(I,J)^2}|I|_\sigma$, trivially valid when $3I\cap J=\emptyset$ and $|J|\leq|I|$. We may assume that $\|f\|_{L^2(\sigma)}^2=\|g\|_{L^2(\omega)}^2=1$. We then estimate

$$\begin{aligned}
A_{long-range} &\leq \sum_{I\in\mathcal{F}}\sum_{J\,:\,|J|\leq|I|:dist(I,J)\geq|I|}|\mathbb{E}_I^\sigma f|\beta(I,J)|\langle g,h_J^\omega\rangle_\omega| \\
&\lesssim \sum_{I\in\mathcal{F}}\sum_{J\,:\,|J|\leq|I|:dist(I,J)\geq|I|}|\mathbb{E}_I^\sigma f|\sqrt{|I|_\sigma}\sqrt{|I|_\sigma}\frac{|J|}{dist(I,J)^2}\sqrt{|J|_\omega}|\langle g,h_J^\omega\rangle_\omega| \\
&\lesssim \sum_{I\in\mathcal{F}}|\mathbb{E}_I^\sigma f|^2|I|_\sigma\sum_{J\,:\,|J|\leq|I|:dist(I,J)\geq|I|}\left(\frac{|J|}{|I|}\right)^\delta\sqrt{|I|_\sigma}\frac{|J|}{dist(I,J)^2}\sqrt{|J|_\omega} \\
&\quad+\sum_J|\langle g,h_J^\omega\rangle_\omega|^2\sum_{I\,:\,|J|\leq|I|:dist(I,J)\geq|I|}\left(\frac{|J|}{|I|}\right)^{-\delta}\sqrt{|I|_\sigma}\frac{|J|}{dist(I,J)^2}\sqrt{|J|_\omega},
\end{aligned}$$

where we have inserted the gain and loss factors $\left(\frac{|J|}{|I|}\right)^{\pm\delta}$ with $0<\delta<1$ to facilitate application of Schur's test. For each fixed $I\in\mathcal{F}$ we have

$$\begin{aligned}
&\sum_{J\,:\,|J|\leq|I|:dist(I,J)\geq|I|}\left(\frac{|J|}{|I|}\right)^\delta\sqrt{|I|_\sigma}\frac{|J|}{dist(I,J)^2}\sqrt{|J|_\omega} \\
\lesssim\ &\sqrt{|I|_\sigma}\sum_{k=0}^\infty 2^{-k\delta}\left(\sum_{J\,:\,2^k|J|=|I|:dist(I,J)\geq|I|}\frac{|J|}{dist(I,J)^2}|J|_\omega\right)^{\frac12} \\
&\times\left(\sum_{J\,:\,2^k|J|=|I|:dist(I,J)\geq|I|}\frac{|J|}{dist(I,J)^2}\right)^{\frac12},
\end{aligned}$$

which is bounded by

$$C\sum_{k=0}^\infty 2^{-k\delta}\left(\frac{|I|_\sigma}{|I|}\mathsf{P}(I,\omega)\right)^{\frac12}\lesssim\sqrt{\mathcal{A}_2},$$

if $\delta>0$. For each fixed J we have

$$\begin{aligned}
&\sum_{I\,:\,|J|\leq|I|:dist(I,J)\geq|I|}\left(\frac{|J|}{|I|}\right)^{-\delta}\sqrt{|I|_\sigma}\frac{|J|}{dist(I,J)^2}\sqrt{|J|_\omega} \\
\lesssim\ &\sqrt{|J|_\omega}\sum_{k=0}^\infty 2^{-k(1-\delta)}\sum_{I\,:\,2^k|J|=|I|:dist(I,J)\geq|I|}\frac{|I|}{dist(I,J)^2}\sqrt{|I|_\sigma} \\
\lesssim\ &\sqrt{|J|_\omega}\sum_{k=0}^\infty 2^{-k(1-\delta)}\left(\sum_{I\,:\,2^k|J|=|I|:dist(I,J)\geq|I|}\frac{|I|}{dist(I,J)^2}|I|_\sigma\right)^{\frac12} \\
&\times\left(\sum_{I\,:\,2^k|J|=|I|:dist(I,J)\geq|I|}\frac{|I|}{dist(I,J)^2}\right)^{\frac12},
\end{aligned}$$

which is bounded by

$$\begin{aligned} &\sqrt{|J|_\omega}\sum_{k=0}^{\infty}2^{-k(1-\delta)}\mathsf{P}\left(2^kJ,\sigma\right)^{\frac12}\left(\frac{1}{|2^kJ|}\right)^{\frac12} \\ \lesssim &\sum_{k=0}^{\infty}2^{-k(1-\delta)}\left(\frac{\left|2^kJ\right|_\omega}{|2^kJ|}\mathsf{P}\left(2^kJ,\sigma\right)\right)^{\frac12}\lesssim\sqrt{\mathcal{A}_2}, \end{aligned}$$

if $\delta<1$. With any fixed $0<\delta<1$ we obtain from the inequalities above that

$$\begin{aligned} A_{long-range} &\lesssim \sum_{I\in\mathcal{F}}\left|\mathbb{E}_I^\sigma f\right|^2|I|_\sigma\sqrt{\mathcal{A}_2}+\sum_J|\langle g,h_J^\omega\rangle_\omega|^2\sqrt{\mathcal{A}_2} \\ &\lesssim \left(\|f\|_{L^2(\sigma)}^2+\|\phi\|_{L^2(\omega)}^2\right)\sqrt{\mathcal{A}_2}=2\sqrt{\mathcal{A}_2}\,\|f\|_{L^2(\sigma)}\,\|g\|_{L^2(\omega)}\,, \end{aligned}$$

since we assumed $\|f\|_{L^2(\sigma)}=\|g\|_{L^2(\omega)}=1$, and this completes the proof of (3.18).

Now we turn to the proof for the mid-range sum, namely we prove

$$\begin{aligned} A_{mid-range} &\equiv \sum_{\substack{(I,J)\in\mathcal{F}\times\mathcal{D}^\omega\\ J\subset 3I\setminus I\text{ and }|J|\leq 2^{-r}|I|}}\left|\left\langle H_\sigma\left(\beta_{I,J}\mathbf{1}_I\mathbb{E}_I^\sigma f\right),\triangle_J^\omega g\right\rangle_\omega\right| \\ &\lesssim \sqrt{\mathcal{A}_2}\,\|f\|_{L^2(\sigma)}\,\|g\|_{L^2(\omega)}\,. \end{aligned} \tag{3.19}$$

To see (3.19), we set for integers $s\geq r$,

$$\begin{aligned} A_{mid-range}(s) &\equiv \sum_{I\in\mathcal{F}}\sum_{J\,:\,2^s|J|=|I|:dist(I,J)\leq|I|\,,\ I\cap J=\emptyset}\left|\left(\mathbb{E}_I^\sigma|f|\right)\left\langle H_\sigma(\beta_{I,J}\mathbf{1}_I),h_J^\omega\right\rangle_\omega\langle g,h_J^\omega\rangle_\omega\right| \\ &\lesssim \left[\sum_{I\in\mathcal{F}}\left(\mathbb{E}_I^\sigma|f|\right)^2|I|_\sigma\right]^{\frac12} \\ &\quad\times\left[\sum_{I\in\mathcal{F}}\left(\sum_{J\,:\,2^s|J|=|I|:dist(I,J)\leq|I|\,,\ I\cap J=\emptyset}|I|_\sigma^{-\frac12}\left|\left\langle H_\sigma(\beta_{I,J}\mathbf{1}_I),h_J^\omega\right\rangle_\omega\right|\left|\langle g,h_J^\omega\rangle_\omega\right|\right)^2\right]^{1/2} \\ &\lesssim \|f\|_{L^2(\sigma)}\left[2^s\sum_{I\in\mathcal{F}}\sum_{J\,:\,2^s|J|=|I|:dist(I,J)\leq|I|\,,\ I\cap J=\emptyset}|I|_\sigma^{-1}\left|\left\langle H_\sigma(\beta_{I,J}\mathbf{1}_I),h_J^\omega\right\rangle_\omega\right|^2\left|\langle g,h_J^\omega\rangle_\omega\right|^2\right]^{1/2} \\ &\lesssim \Lambda(s)\,\|f\|_{L^2(\sigma)}\,\|g\|_{L^2(\omega)}\,, \end{aligned}$$

where

$$\begin{aligned} \Lambda(s)^2 &\equiv 2^s\sup_{I\in\mathcal{F}}\sup_{J\,:\,2^s|J|=|I|:dist(I,J)\leq|I|\,,\ I\cap J=\emptyset}|I|_\sigma^{-1}\left|\left\langle H_\sigma(\beta_{I,J}\mathbf{1}_I),h_J^\omega\right\rangle_\omega\right|^2 \\ &\leq 2^s\sup_{I\in\mathcal{F}}\sum_{J\,:\,2^s|J|=|I|:dist(I,J)\leq|I|\,,\ I\cap J=\emptyset}|I|_\sigma^{-1}\left|\left\langle H_\sigma(\beta_{I,J}\mathbf{1}_I),h_J^\omega\right\rangle_\omega\right|^2, \end{aligned}$$

since

$$\sum_{I\in\mathcal{F}}\sum_{J\,:\,2^s|J|=|I|:dist(I,J)\leq|I|\,,\ I\cap J=\emptyset}\left|\langle g,h_J^\omega\rangle_\omega\right|^2\leq 2\sum_J\left|\langle g,h_J^\omega\rangle_\omega\right|^2=2\,\|g\|_{L^2(\omega)}^2\ .$$

Due to the 'local' nature of the sum in J, we have thus gained a small improvement in the Schur test to derive the last line.

But (3.10) applies since $2J$ is disjoint from the support of $\beta_{I,J}$, J is good with respect to the grid $\mathcal{D}^\sigma$, and so (3.15) also applies to yield

$$\begin{aligned}
\Lambda(s)^2 &\lesssim \sup_{I\in\mathcal{F}} 2^s \sum_{J\,:\,2^s|J|=|I|:dist(I,J)\leq|I|\,,\ I\cap J=\emptyset} \frac{\omega(J)}{\sigma(I)} \cdot \mathsf{P}(J,\mathbf{1}_I\sigma)^2 \\
&\lesssim \sup_{I\in\mathcal{F}} 2^s \sum_{J\,:\,2^s|J|=|I|:dist(I,J)\leq|I|\,,\ I\cap J=\emptyset} \frac{\omega(J)}{\sigma(I)} \left(\frac{|J|}{|I|}\right)^{2-4\varepsilon} \cdot \mathsf{P}(I,\mathbf{1}_I\sigma)^2 \\
&\lesssim \sup_{I\in\mathcal{F}} 2^s 2^{-s(2-4\varepsilon)} \frac{\sigma(I)}{|I|^2} \sum_{J\,:\,2^s|J|=|I|:dist(I,J)\leq|I|\,,\ I\cap J=\emptyset} \omega(J) \\
&\lesssim 2^{-(1-4\varepsilon)s}\mathcal{A}_2\,.
\end{aligned}$$

This is clearly a summable estimate in $s\geq r$, so the proof of (3.19) is complete. □

3.3. **The good-bad decomposition.** Here we use the random grid idea of Nazarov, Treil and Volberg (see e.g. Chapter 17 of [Vol]) as more recently refined in Hytönen, Pérez, Treil and Volberg [1]. For any $\beta=\{\beta_l\}\in\{0,1\}^{\mathbb{Z}}$, define the dyadic grid $\mathbb{D}_\beta$ to be the collection of intervals

$$\mathbb{D}_\beta=\left\{2^n\left([0,1)+k+\sum_{i<n}2^{i-n}\beta_i\right)\right\}_{n\in\mathbb{Z},\,k\in\mathbb{Z}}$$

This parametrization of dyadic grids appears explicitly in [Hyt], and implicitly in [NTV2] section 9.1. Place the usual uniform probability measure $\mathbb{P}$ on the space $\{0,1\}^{\mathbb{Z}}$, explicitly

$$\mathbb{P}(\beta:\beta_l=0)=\mathbb{P}(\beta:\beta_l=1)=\frac{1}{2},\qquad \text{for all } l\in\mathbb{Z},$$

and then extend by independence of the β_l. Note that the *endpoints* and *centers* of the intervals in the grid $\mathbb{D}_\beta$ are contained in $\mathbb{Q}^{dy}+x_\beta$, the dyadic rationals $\mathbb{Q}^{dy}\equiv\left\{\frac{m}{2^n}\right\}_{m,n\in\mathbb{Z}}$ translated by $x_\beta\equiv\sum_{i<0}2^i\beta_i\in[0,1]$. Moreover the pushforward of the probability measure $\mathbb{P}$ under the map $\beta\to x_\beta$ is Lebesgue measure on $[0,1]$. The locally finite weights ω,σ have at most countably many point masses, and it follows with probability one that ω,σ do *not* charge an endpoint or center of any interval in $\mathbb{D}_\beta$.

For a weight ω, we consider a random choice of dyadic grid $\mathcal{D}^\omega$ on the probability space Σ^ω, and likewise for second weight σ, with a random choice of dyadic grid $\mathcal{D}^\sigma$ on the probability space Σ^σ.

Notation 3.20. *We fix $\varepsilon>0$ for use throughout the remainder of the paper.*

Definition 3.21. *For a positive integer r, an interval $J\in\mathcal{D}^\sigma$ is said to be* r-bad *if there is an interval $I\in\mathcal{D}^\omega$ with $|I|\geq 2^r|I|$, and*

$$\operatorname{dist}(e(I),J)\leq\tfrac{1}{2}|J|^\varepsilon|I|^{1-\varepsilon}\,.$$

Here, $e(J)$ is the set of three points consisting of the two endpoints of J and its center. (This is the set of discontinuities of h_J^σ.) Otherwise, J is said to be r-good. *We symmetrically define $J\in\mathcal{D}^\omega$ to be* r-good.

Let $\mathcal{D}^\sigma$ be randomly selected, with parameter β, and $\mathcal{D}^\omega$ with parameter β'. Define a projection

$$\mathsf{P}^\sigma_{good} f \equiv \sum_{I \text{ is } r\text{-good } \in \mathcal{D}^\sigma} \Delta^\sigma_I f\,, \tag{3.22}$$

and likewise for $\mathsf{P}^\omega_{good} g$. Define an r,ε-good *subgrid* $\mathcal{D}^\sigma_{good}$ by

$$\mathcal{D}^\sigma_{good} = \left\{ I \in \mathcal{D}^\sigma : dist\left(I, e\left(\widehat{I}\right)\right) \geq |I|^\varepsilon \left|\widehat{I}\right|^{1-\varepsilon} \text{ whenever } \left|\widehat{I}\right| \geq 2^r |I| \right\}. \tag{3.23}$$

Now define an r,ε-good projection $\mathsf{G}_{\mathcal{D}^\sigma}$ on $L^2(\sigma)$ by

$$\mathsf{G}_{\mathcal{D}^\sigma} f = \sum_{I \in \mathcal{D}^\sigma_{good}} \langle f, h^\sigma_I \rangle_\sigma h^\sigma_I, \qquad f \in L^2(\sigma)\,. \tag{3.24}$$

Let $T : L^2(\sigma) \to L^2(\omega)$ be a bounded linear operator. Then the operator norm $\|T\|_{L^2(\sigma) \longrightarrow L^2(\omega)}$ is bounded by a multiple of

$$\mathbb{E}_\beta \mathbb{E}_{\beta'} \sup_{\|f\|_{L^2(\sigma)} = \|g\|_{L^2(\omega)} = 1} \left| \left\langle T \mathsf{P}^\sigma_{good} \mathsf{G}_{\mathcal{D}^\sigma} f, \mathsf{P}^\omega_{good} \mathsf{G}_{\mathcal{D}^\omega} g \right\rangle_\omega \right|,$$

where $\mathbb{E}_\beta, \mathbb{E}_{\beta'}$ are the expectations relative to the probability space of grids, the projections P^σ_{good} and P^ω_{good} are defined in (3.22), and the projections $\mathsf{G}_{\mathcal{D}^\sigma}$ and $\mathsf{G}_{\mathcal{D}^\omega}$ are defined in (3.24). The constant $\varepsilon > 0$ in these definitions is taken sufficiently small, and the associated constant $r > 0$ is then taken sufficiently large depending on ε.

Summary 3.25. *It suffices to consider only r-good intervals, and only functions of the form $f = G_{\mathcal{D}^\sigma} f$ and $g = G_{\mathcal{D}^\omega} g$, and prove an estimate for $\|H(\sigma\cdot)\|_{L^2(\sigma)\to L^2(\omega)}$ that is independent of these assumptions. Accordingly, we will call r-good intervals just* good intervals *from now on, and we will assume $f = \mathsf{G}_{\mathcal{D}^\sigma} f$ and $g = \mathsf{G}_{\mathcal{D}^\omega} g$. It is important to note that an interval $J \in \mathcal{D}^\omega_{good}$ satisfies the 'good' inequality (3.23) with respect to* both *grids $\mathcal{D}^\sigma$ and $\mathcal{D}^\omega$. In fact, we will assume that the two grids $\mathcal{D}^\sigma$ and $\mathcal{D}^\omega$ actually coincide. For this see Hytönen, Pérez, Treil and Volberg [1].*

Remark 3.26. *If $J \subset I$ is an r,ε-good subinterval of an interval I, then one of the following two cases holds:*
Case (1): *either $|J| > 2^{-r} |I|$,*
Case (2): *or $|J| \leq 2^{-r} |I|$ and $dist(J, e(I)) \geq |J|^\varepsilon |I|^{1-\varepsilon}$.*
For a fixed interval I, there are only 2^{r+1} intervals J in Case (1), and since the lengths of $|J|$ and $|I|$ are comparable, all of the estimates we claim in this paper for Case (1) subintervals J turn out to be essentially trivial. Thus the Case (2) subintervals J constitute the substantial case, and for these subintervals we write $J \Subset I$ and refer to J as simply a good *subinterval of I.*

4. Intertwining proposition and functional energy

Our main result here says that, modulo terms that are controlled by the $\mathcal{A}_2$ and interval testing conditions, we can in two special situations, pass the ω-corona projection $\mathsf{P}^\omega_{\mathcal{C}^\omega_G}$ through the Hilbert transform H to become the σ-corona projection $\mathsf{P}^\sigma_{\mathcal{C}^\omega_G}$. More precisely, we mean that with $H_\sigma f \equiv H(f\sigma)$, the intertwining operator

$$\mathsf{P}^\omega_{\mathcal{C}^\omega_G} \left[\mathsf{P}^\omega_{\mathcal{C}^\omega_G} H_\sigma - H_\sigma \mathsf{P}^\sigma_{\mathcal{C}^\omega_G} \right] \mathsf{P}^\sigma_{\mathcal{C}^\sigma_F}$$

is bounded with constant $\mathcal{NTV}$. The first special situation in which this works is when $G\subset F$ and $(F,G)\in\mathsf{Far}(\mathcal{F}\times\mathcal{G})$, in which case the intertwining operator reduces to $\mathsf{P}^{\omega}_{\mathcal{C}^{\omega}_G}H_\sigma\mathsf{P}^{\sigma}_{\mathcal{C}_F}$ since $\mathcal{C}^{\sigma}_F\cap\mathcal{C}^{\omega}_G=\emptyset$.

The case when $(F,G)\in\mathsf{Near}(\mathcal{F}\times\mathcal{G})$ is more problematic, and we do not know if the analogous result holds for it. However, we *can* pass the ω-corona projection $\mathsf{P}^{\omega}_{\mathcal{C}^{\omega}_G}$ through a restricted portion of the sum, which we now describe. For a fixed $F\in\mathcal{F}$, it will be convenient to write $G\sim F$ to mean that G satisfies the properties $(F,G)\in\mathsf{Near}(\mathcal{F}\times\mathcal{G})$ and $G\subset F$, so that we can iterate the near sum as

$$\sum_{\substack{(F,G)\in\mathsf{Near}(\mathcal{F}\times\mathcal{G})\\ G\subset F}}\left\{\sum_{(I,J)\in\mathcal{C}^{\sigma}_F\times\mathcal{C}^{\omega}_G}\left\langle H_\sigma\bigtriangleup^{\sigma}_I f,\bigtriangleup^{\omega}_J g\right\rangle_\omega\right\}$$
$$=\sum_{F\in\mathcal{F}}\sum_{G\sim F}\left\{\sum_{(I,J)\in\mathcal{C}^{\sigma}_F\times\mathcal{C}^{\omega}_G}\left\langle H_\sigma\bigtriangleup^{\sigma}_I f,\bigtriangleup^{\omega}_J g\right\rangle_\omega\right\}.$$

The additional restriction we impose on the inner sum above is that I is contained in a $\mathcal{G}$-child G' of G; thus while $G\subset F$, there is a sort of 'reverse' inclusion required of I. We will establish an intertwining inequality for the 'mixed' form,

$$\mathsf{B}_{mix}(f,g)\equiv\sum_{F\in\mathcal{F}}\sum_{G\sim F}\sum_{\substack{(I,J)\in\mathcal{C}^{\sigma}_F\times\mathcal{C}^{\omega}_G\\ I\Subset G'\in\mathfrak{C}_{\mathcal{G}}(G)}}\left\langle\bigtriangleup^{\sigma}_I f,H_\omega\bigtriangleup^{\omega}_J g\right\rangle_\sigma. \tag{4.1}$$

We separate one of the main steps in the proof in the second subsection below, where we show in the Functional Energy Proposition 4.22 that the functional energy conditions are controlled by the $\mathcal{A}_2$ and interval testing conditions. We will freely apply the Intertwining Proposition 4.2 in the more complicated analysis of bounded fluctuation in later sections.

4.1. Intertwining proposition.

Proposition 4.2. *Let $f=\sum_{F\in\mathcal{F}}\mathsf{P}^{\sigma}_{\mathcal{C}^{\sigma}_F}f$ and $g=\sum_{G\in\mathcal{G}}\mathsf{P}^{\omega}_{\mathcal{C}^{\omega}_G}g$ be a parallel Calderón-Zygmund corona decomposition for $f\in L^2(\sigma)$ and $g\in L^2(\omega)$. Then*

$$\left|\sum_{\substack{(F,G)\in\mathsf{Far}(\mathcal{F}\times\mathcal{G})\\ G\subset F}}\left\langle H_\sigma\mathsf{P}^{\sigma}_{\mathcal{C}^{\sigma}_F}f,\mathsf{P}^{\omega}_{\mathcal{C}^{\omega}_G}g\right\rangle_\omega\right|+\left|\mathsf{B}_{mix}(f,g)\right|\lesssim(\mathfrak{NTV})\left\|f\right\|_{L^2(\sigma)}\left\|g\right\|_{L^2(\omega)}. \tag{4.3}$$

There is of course a dual formulation with $F\subset G$.

We will also need the following generalization of the Intertwining Proposition to parallel corona decompositions that use general stopping data as defined in Definition 1.8.

Proposition 4.4. *Let*

$$\left\langle H_\sigma(f),g\right\rangle_\omega=\mathsf{H}_{near}(f,g)+\mathsf{H}_{disjoint}(f,g)+\mathsf{H}_{far}(f,g)$$

be a parallel corona decomposition as in (1.7) of the bilinear form $\left\langle H_\sigma f,g\right\rangle_\omega$ with stopping data $\mathcal{F}$ and $\mathcal{G}$ for f and g respectively. Then (4.3) holds.

In this first subsection we prove that the left hand side of (4.3) is dominated by the larger expression

$$(4.5) \qquad \left(\mathfrak{NTV}+\mathfrak{F}+\mathfrak{F}^{*}\right)\left\|f\right\|_{L^{2}(\sigma)}\left\|g\right\|_{L^{2}(\omega)},$$

that includes the functional energy constants $\mathfrak{F}$ and $\mathfrak{F}^{*}$. In the next subsection we show that the functional energy constants $\mathfrak{F}$ and $\mathfrak{F}^{*}$ are themselves controlled by $\mathfrak{NTV}$. Finally, in the third subsection we prove the more general Proposition 4.4.

Remark 4.6. *We do not know if the following intertwining inequality holds:*

$$\left|\sum_{\substack{(F,G)\in\mathsf{Near}(\mathcal{F}\times\mathcal{G})\\ G\subset F}}\left\langle H_{\sigma}\mathsf{P}^{\sigma}_{\mathcal{C}^{\sigma}_{F}\setminus\mathcal{C}^{\omega}_{G}}f,\mathsf{P}^{\omega}_{\mathcal{C}^{\omega}_{G}}g\right\rangle_{\omega}\right|\lesssim\left(\mathfrak{NTV}\right)\left\|f\right\|_{L^{2}(\sigma)}\left\|g\right\|_{L^{2}(\omega)}\ .$$

If true, we could use the techniques of this paper to replace the indicator/interval testing conditions (1.3) with (1.2) and the following weaker inequality and its dual: $\left|\left\langle H_{\sigma}\mathbf{1}_{E},\mathbf{1}_{E'}\right\rangle_{\omega}\right|\lesssim\sqrt{\left|I\right|_{\sigma}\left|E'\right|_{\omega}}$ *for all intervals I and compact subsets $E,E'\subset I$.*

Proof. We prove the inequality (4.3) with (4.5) on the right side. We begin by writing

$$\begin{aligned}\mathsf{H}_{far\ lower}\left(f,g\right) &= \sum_{\substack{(F,G)\in\mathsf{Far}(\mathcal{F}\times\mathcal{G})\\ G\subset F}}\left\langle H_{\sigma}\mathsf{P}^{\sigma}_{\mathcal{C}^{\sigma}_{F}}f,\mathsf{P}^{\omega}_{\mathcal{C}^{\omega}_{G}}g\right\rangle_{\omega}\\ &= \sum_{J\in\mathcal{D}^{\omega}}\sum_{\substack{(F,G)\in\mathsf{Far}(\mathcal{F}\times\mathcal{G})\\ J\in\mathcal{C}_{G}\text{ and }G\subset F}}\left\langle H_{\sigma}\mathsf{P}^{\sigma}_{\mathcal{C}_{F}}f,\triangle^{\omega}_{J}g\right\rangle_{\omega}\ ,\end{aligned}$$

and claim the estimate

$$(4.7) \qquad \left|\mathsf{H}_{far\ lower}\left(f,g\right)\right|\lesssim\left(\mathfrak{NTV}+\mathfrak{F}\right)\left\|f\right\|_{L^{2}(\sigma)}\left\|g\right\|_{L^{2}(\omega)}.$$

We have

$$\begin{aligned}\mathsf{H}_{far\ lower}\left(f,g\right) &= \sum_{\substack{(F,G)\in\mathsf{Far}(\mathcal{F}\times\mathcal{G})\\ G\subset F}}\int H_{\sigma}\left(\mathsf{P}^{\sigma}_{\mathcal{C}_{F}}f\right)\ \left(\mathsf{P}^{\omega}_{\mathcal{C}_{G}}g\right)\omega\\ &= \sum_{G\in\mathcal{G}}\int H_{\sigma}\left(\sum_{\substack{F\in\mathcal{F}:\ (F,G)\in\mathsf{Far}(\mathcal{F}\times\mathcal{G})\\ G\subset F}}\mathsf{P}^{\sigma}_{\mathcal{C}_{F}}f\right)\ \left(\mathsf{P}^{\omega}_{\mathcal{C}_{G}}g\right)\omega\\ &\equiv \sum_{G\in\mathcal{G}}\int H_{\sigma}\left(\widehat{f}_{G}\right)\ \left(\mathsf{P}^{\omega}_{\mathcal{C}_{G}}g\right)\omega,\end{aligned}$$

where

$$\widehat{f}_{G}\equiv\sum_{\substack{F\in\mathcal{F}:\ (F,G)\in\mathsf{Far}(\mathcal{F}\times\mathcal{G})\\ G\subset F}}\mathsf{P}^{\sigma}_{\mathcal{C}_{F}}f=\sum_{\substack{F\in\mathcal{F}:\ (F,G)\in\mathsf{Far}(\mathcal{F}\times\mathcal{G})\\ G\subset F}}\sum_{I\in\mathcal{C}_{F}}\triangle^{\sigma}_{I}f,\qquad G\in\mathcal{G}.$$

Now we decompose this last sum according to whether or not the interval I is disjoint from G:

$$\begin{aligned}
\widehat{f}_G &= f_G^{\natural} + f_G^{\flat}; \\
f_G^{\natural} &= \sum_{\substack{F\in\mathcal{F}:\ (F,G)\in\mathsf{Far}(\mathcal{F}\times\mathcal{G}) \\ G\subset F}} \sum_{I\in\mathcal{C}_F:\ G\subset I} \triangle_I^{\sigma} f, \\
f_G^{\flat} &= \sum_{\substack{F\in\mathcal{F}:\ (F,G)\in\mathsf{Far}(\mathcal{F}\times\mathcal{G}) \\ G\subset F}} \sum_{I\in\mathcal{C}_F:\ G\cap I=\emptyset} \triangle_I^{\sigma} f.
\end{aligned}$$

Once again we have

$$\begin{aligned}
\left|\sum_{G\in\mathcal{G}} \int H_{\sigma}\left(f_G^{\flat}\right)\ \left(\mathsf{P}_{\mathcal{C}_G}^{\omega} g\right)\omega\right| &= \left|\sum_{G\in\mathcal{G}} \sum_{\substack{F\in\mathcal{F}:\ (F,G)\in\mathsf{Far}(\mathcal{F}\times\mathcal{G}) \\ G\subset F}} \sum_{\substack{I\in\mathcal{C}_F \\ I\cap G=\emptyset}} \sum_{J\in\mathcal{C}_G} \int H_{\sigma}\left(\triangle_I^{\sigma} f\right)\ \left(\triangle_J^{\omega} g\right)\omega\right| \\
&\leq \sum_{(I,J)\in\mathcal{D}^{\sigma}\times\mathcal{D}^{\omega}:\ I\cap J=\emptyset} \left|\left\langle H_{\sigma}\left(\triangle_I^{\sigma} f\right), \triangle_J^{\omega} g\right\rangle_{\omega}\right| \\
&\lesssim \left(\mathfrak{NTV}\right) \|f\|_{L^2(\sigma)} \|g\|_{L^2(\omega)},
\end{aligned}$$

by Lemma 3.1 since the intervals I and J paired above are disjoint.

Turning to the term involving $f_G^{\natural}$, we note that the intervals I occurring there are linearly and consecutively ordered by inclusion, along with the intervals $F\in\mathcal{F}$ that contain G. More precisely, we can write

$$G\subset F_1(G) \subsetneqq F_2(G) \subsetneqq \dots F_n(G) \subsetneqq F_{n+1}(G) \subsetneqq \dots F_N(G)$$

where $F_1(G)$ is the smallest interval in $\mathcal{F}$ containing G and $F_{n+1}(G)=\pi_{\mathcal{F}}F_n(G)$ for all $n\geq 1$. Note that the only intervals $F_n(G)$ occurring among the intervals I in the sum for $f_G^{\natural}$ are those with $n\geq 2$, since we must have $(F_n(G),G)\in\mathsf{Far}(\mathcal{F}\times\mathcal{G})$, which requires that there is $F'\in\mathcal{F}$ satisfying $G\subset F'\subsetneqq F_n(G)$. We can also write

$$G\subset I_1(G) \subsetneqq I_2(G) \subsetneqq \dots I_k(G) \subsetneqq I_{k+1}(G) \subsetneqq \dots I_K(G) = F_N(G)$$

where $I_1(G)$ is the smallest interval in $\mathcal{C}_{F_2(G)}$ containing G, equivalently $I_1(G)=\pi_{\mathcal{D}^{\sigma}}F_1(G)$, and $I_{k+1}(G)=\pi_{\mathcal{D}^{\sigma}}I_k(G)$ for all $k\geq 1$. There is a (unique) subsequence $\{k_n\}_{n=1}^N$ such that

$$F_n(G) = I_{k_n}(G), \qquad 1\leq n\leq N,$$

upon defining $I_{k_1}(G)=I_0(G)=F_1(G)$. Note here that the only intervals $I_k(G)$ occurring among the intervals I in the sum for $f_G^{\natural}$ are those with $k>k_1$.

Assume now that $k_n\leq k<k_{n+1}$ and

$$\theta\left(I_k(G)\right) = I_{k+1}(G)\setminus I_k(G)\in\mathcal{C}_{F_{n+1}}. \tag{4.8}$$

Then using a telescoping sum, we compute that for $x\in\theta\left(I_k(G)\right)$,

$$\left|f_G^{\natural}(x)\right| = \left|\sum_{\ell=k}^{\infty} \triangle_{I_\ell}^{\sigma} f(x)\right| = \left|\mathbb{E}_{\theta(I_k(G))}^{\sigma} f - \mathbb{E}_{I_K}^{\sigma} f\right| \lesssim \mathbb{E}_{I_{k_{n+1}}}^{\sigma}|f| = \mathbb{E}_{F_{n+1}}^{\sigma}|f|. \tag{4.9}$$

If $k_n\leq k<k_{n+1}$ and (4.8) fails, then we have

$$\theta\left(I_k(G)\right)\in\mathfrak{C}\left(F_{n+1}\right)\subset\mathcal{F}.$$

Thus we decompose $f_G^\natural(x)$ as (recall $I_{k_1}(G)=F_1(G)$)

$$f_G^\natural = f_{G,local}^\natural + f_{G,corona}^\natural + f_{G,stopping}^\natural ,$$

where

$$\begin{aligned} f_{G,local}^\natural &= \left(\mathbb{E}_{F_1(G)}^\sigma f - \mathbb{E}_{I_K}^\sigma f\right)\mathbf{1}_{F_1(G)} , \\ f_{G,corona}^\natural &= \sum_{k\geq k_1:\ \theta(I_k(G))\notin\mathcal{F}} \left(\mathbb{E}_{\theta(I_k(G))}^\sigma f - \mathbb{E}_{I_K}^\sigma f\right)\mathbf{1}_{\theta(I_k(G))} , \\ f_{G,stopping}^\natural &= \sum_{k\geq k_1:\ \theta(I_k(G))\in\mathcal{F}} \left(\mathbb{E}_{\theta(I_k(G))}^\sigma f - \mathbb{E}_{I_K}^\sigma f\right)\mathbf{1}_{\theta(I_k(G))} . \end{aligned}$$

Now $f_{G,local}^\natural$ depends only on $F_1(G)$, and $\left|f_{G,local}^\natural\right| \lesssim \left(\mathbb{E}_{F_1(G)}^\sigma |f|\right)\mathbf{1}_{F_1(G)}$, so that if we write

$$\begin{aligned} f_{F_1(G)}^{@} &\equiv f_{G,local}^\natural, \\ \mathsf{R}_F^\omega g &\equiv \sum_{G\in\mathcal{G}:\ F_1(G)=F} \mathsf{P}_{\mathcal{C}_G}^\omega g, \end{aligned}$$

then we have

$$\begin{aligned} (4.10)\qquad & \left|\sum_{G\in\mathcal{G}}\int H_\sigma\left(f_{G,local}^\natural\right)\left(\mathsf{P}_{\mathcal{C}_G}^\omega g\right)\omega\right| = \left|\sum_{F\in\mathcal{F}}\int H_\sigma\left(f_F^{@}\right)\left(\mathsf{R}_F^\omega g\right)\omega\right| \\ &\lesssim \mathfrak{T}\sum_{F\in\mathcal{F}}\left(\mathbb{E}_F^\sigma|f|\right)\sqrt{|F|_\sigma}\left\|\mathsf{R}_F^\omega g\right\|_{L^2(\omega)} \\ &\lesssim \mathfrak{T}\left(\sum_{F\in\mathcal{F}}\left(\mathbb{E}_F^\sigma|f|\right)^2|F|_\sigma\right)^{\frac12}\left(\sum_{F\in\mathcal{F}}\left\|\mathsf{R}_F^\omega g\right\|_{L^2(\omega)}^2\right)^{\frac12} \\ &\lesssim \mathfrak{T}\left\|f\right\|_{L^2(\sigma)}\left\|g\right\|_{L^2(\omega)} . \end{aligned}$$

Next we turn to estimating $f_{G,stopping}^\natural$ in the decomposition of $f_G^\natural$, which can be controlled by the $\mathcal{A}_2$ condition alone. We claim

$$(4.11)\qquad \left|\sum_{G\in\mathcal{G}}\int H_\sigma\left(f_{G,stopping}^\natural\right)\left(\mathsf{P}_{\mathcal{C}_G}^\omega g\right)\omega\right| \lesssim \sqrt{\mathcal{A}_2}\left\|f\right\|_{L^2(\sigma)}\left\|g\right\|_{L^2(\omega)} .$$

To prove this we write

$$\begin{aligned} (4.12)\qquad & \left|\sum_{G\in\mathcal{G}}\int H_\sigma\left(f_{G,stopping}^\natural\right)\left(\mathsf{P}_{\mathcal{C}_G}^\omega g\right)\omega\right| \\ &= \left|\sum_{G\in\mathcal{G}}\sum_{F\in\mathcal{F}:\ G\subset\theta(F)}\left(\mathbb{E}_F^\sigma f\right)\left\langle H_\sigma\left(\mathbf{1}_F\right),\mathsf{P}_{\mathcal{C}_G}^\omega g\right\rangle_\omega\right| \\ &= \left|\sum_{F\in\mathcal{F}}\left(\mathbb{E}_F^\sigma f\right)\left\langle H_\sigma\left(\mathbf{1}_F\right),\sum_{G\in\mathcal{G}:\ G\subset\text{some }F'\subset\theta(F)}\mathsf{P}_{\mathcal{C}_G}^\omega g\right\rangle_\omega\right| \\ &\leq \sum_{F\in\mathcal{F}}\left(\mathbb{E}_F^\sigma f\right)\sum_{J\subset\theta(F)}\left|\left\langle H_\sigma\left(\mathbf{1}_F\right),\triangle_J^\omega g\right\rangle_\omega\right| . \end{aligned}$$

Lemma 3.16 applies to the final line above to give (4.11). We remark that using $J\subset\theta(F)$ in the final sum, we can replace $\sqrt{\mathcal{A}_2}$ with the classical constant $\sqrt{A_2}$ in the estimate above.

To handle $f^{\natural}_{G,corona}$ we will use Lemma 3.2 and the functional energy condition (2.6) above in conjunction with the representation

$$f^{\natural}_{G,corona}=\sum_{k\geq k_1:\ \theta(I_k(G))\notin\mathcal{F}}\left(\mathbb{E}^{\sigma}_{\theta(I_k(G))}f-\mathbb{E}^{\sigma}_{I_K}f\right)\mathbf{1}_{\theta(I_k(G))}=\sum_{n=1}^{N}\beta_{F_{n+1},G}\mathbf{1}_{F_{n+1}},$$

where the function $\beta_{F_{n+1},G}$ defined by

$$\beta_{F_{n+1},G}\equiv\sum_{k_n\leq k<k_{n+1}:\ \theta(I_k(G))\notin\mathcal{F}}\left(\mathbb{E}^{\sigma}_{\theta(I_k(G))}f-\mathbb{E}^{\sigma}_{I_K}f\right)\mathbf{1}_{\theta(I_k(G))},$$

has support in $F_{n+1}\setminus F_n$. Moreover, $\beta_{F_{n+1},G}$ satisfies the following pointwise estimate by (4.9):

$$\left|\beta_{F_{n+1},G}(x)\right|\leq\left(\mathbb{E}^{\sigma}_{F_{n+1}}|f|\right)\mathbf{1}_{F_{n+1}\setminus F_n}(x).\tag{4.13}$$

Thus with $G(J)=G$ for $J\in\mathcal{C}_G$, we can write

$$\sum_{G\in\mathcal{G}}\int H_\sigma\left(f^{\natural}_{G,corona}\right)\left(\mathsf{P}^{\omega}_{\mathcal{C}_G}g\right)\omega=\sum_{J\in\mathcal{D}^\omega}\int H_\sigma\left(f^{\natural}_{G(J),corona}\right)\left(\triangle^{\omega}_J g\right)\omega,$$

and then by the Monotonicity Lemma 3.2 and the bound (4.13) we have

$$\left|\widehat{H_\sigma f^{\natural}_{G(J),corona}}(J)\right|\leq\sum_{\substack{F\in\mathcal{F}:\ G(J)\subset F\\(F,G(J))\in\mathsf{Far}(\mathcal{F}\times\mathcal{G})}}\left\langle H_\sigma\left(\left(\mathbb{E}^{\sigma}_{\pi_{\mathcal{F}}F}|f|\right)\mathbf{1}_{\pi_{\mathcal{F}}F\setminus F}\right),h^{\omega}_J\right\rangle_\omega.$$

Given $J\in\mathcal{D}^\omega$ and $F\in\mathcal{F}$ with $\pi_{\mathcal{F}}J\subset F$, let J^* denote the *maximal* good $\mathcal{D}^\omega$-dyadic interval satisfying $J\subset J^*\subset F$ as in Definition 2.5. Apply the pointwise estimate in the Monotonicity Lemma 3.2 and write

$$\widetilde{g}=\sum_{J\in\mathcal{D}^\omega}\left|\langle g,h^{\omega}_J\rangle_\omega\right|h^{\omega}_J.$$

With $\mathcal{J}(K)$ and $\mathcal{J}^*(K)$ as in Definitions 2.4 and 2.5, we now obtain

$$\left|\sum_{J\in\mathcal{D}^\omega}\int H_\sigma\left(f^{\natural}_{G(J),corona}\right)\left(\triangle^{\omega}_J g\right)\omega\right|\tag{4.14}$$
$$\leq\sum_{J\in\mathcal{D}^\omega}\left|\langle g,h^{\omega}_J\rangle_\omega\right|\sum_{\substack{F\in\mathcal{F}:\ G(J)\subset F\\(F,G(J))\in\mathsf{Far}(\mathcal{F}\times\mathcal{G})}}\left\langle H_\sigma\left(\left(\mathbb{E}^{\sigma}_{\pi_{\mathcal{F}}F}|f|\right)\mathbf{1}_{\pi_{\mathcal{F}}F\setminus F}\right),h^{\omega}_J\right\rangle_\omega$$
$$\lesssim\sum_{J\in\mathcal{D}^\omega}\langle\widetilde{g},h^{\omega}_J\rangle_\omega\sum_{\substack{F\in\mathcal{F}:\ G(J)\subset F\\(F,G(J))\in\mathsf{Far}(\mathcal{F}\times\mathcal{G})}}\left(\mathbb{E}^{\sigma}_{\pi_{\mathcal{F}}F}|f|\right)\frac{1}{|J^*|}\mathsf{P}\left(J^*,\mathbf{1}_{\pi_{\mathcal{F}}F\setminus F}\sigma\right)\ \langle x,h^{\omega}_J\rangle_\omega$$
$$=\sum_{K\in\mathcal{F}}\sum_{J^*\in\mathcal{J}^*(K)}\left\langle x,\sum_{\substack{J\in\mathcal{D}^\omega:\ J\subset J^*\\\pi_{\mathcal{F}}G(J)=K}}\triangle^{\omega}_J\widetilde{g}\right\rangle_\omega\frac{1}{|J^*|}\mathsf{P}\left(J^*,\sum_{F\in\mathcal{F}:\ K\subsetneqq F}\left(\mathbb{E}^{\sigma}_{\pi_{\mathcal{F}}F}|f|\right)\mathbf{1}_{\pi_{\mathcal{F}}F\setminus F}\sigma\right)$$
$$\leq\sum_{K\in\mathcal{F}}\sum_{J^*\in\mathcal{J}^*(K)}\left\langle\frac{x}{|J^*|},\mathbf{1}_{J^*}g_K\right\rangle_\omega\mathsf{P}\left(J^*,\mathcal{M}_\sigma f\right),$$

where the collection of functions

$$g_K \equiv \sum_{\substack{J\in\mathcal{D}^\omega \\ \pi_{\mathcal{F}}G(J)=K}} \triangle_J^\omega \widetilde{g}, \qquad K\in\mathcal{F},$$

is $\mathcal{F}$-adapted as in Definition 2.4 above. Indeed, for $J\in\mathcal{D}^\omega$ and $\pi_{\mathcal{F}}G(J)=K$ we have $\widehat{g_K}(J)=\widehat{\widetilde{g}}(J)\geq 0$, and the orthogonality property

$$\left\langle g_K, g_{K'}\right\rangle_\omega = 0, \qquad K,K'\in\mathcal{F}, \tag{4.15}$$

holds since if $J\in\mathcal{C}_G$, $J'\in\mathcal{C}_{G'}$ and $\pi_{\mathcal{F}}G\neq\pi_{\mathcal{F}}G'$, then $J\neq J'$. Note also that we have the property

$$\left\langle \frac{x}{|J^*|}, \mathbf{1}_{J^*}g_K\right\rangle_\omega = \sum_{\substack{J\in\mathcal{D}^\omega:\ J\subset J^* \\ \pi_{\mathcal{F}}G(J)=K}} \left|\langle g,h_J^\omega\rangle_\omega\right| \left\langle \frac{x}{|J^*|}, h_J^\omega\right\rangle_\omega \geq 0. \tag{4.16}$$

Finally, property (3) of Definition 2.4 holds with overlap constant $C=2$. Indeed, if $J^*\subset I\subset F$ with $J^*\in\mathcal{J}^*(F)$, there are two possibilities: either (i) $G(J^*)\subset I$ or (ii) $I\subsetneqq G(J^*)$. In the first possibility we have $F=F_{G(J^*)}$ and it is now easily seen that the J^* in case (i) are pairwise disjoint. In the second possibility, we have $G(J^*)=G(I)$, and again it is easily seen that the J^* in case (ii) are pairwise disjoint.

Since $\mathcal{F}$ is σ-Carleson, we can now apply the functional energy condition (2.6) to the right side of (4.14) with the choice $h=\mathcal{M}_\sigma f$. We have the maximal function estimate,

$$\|h\|_{L^2(\sigma)} \lesssim \|f\|_{L^2(\sigma)}\,,$$

and so altogether we obtain that the right hand side of (4.14) satisfies

$$\begin{aligned}
&\sum_{K\in\mathcal{F}}\sum_{J^*\in\mathcal{J}^*(K)} \left\langle \frac{x}{|J^*|}, \mathbf{1}_{J^*}g_K\right\rangle_\omega \mathsf{P}\left(J^*,h\sigma\right) \\
\leq\ & \mathfrak{F}\|h\|_{L^2(\sigma)}\left[\sum_{K\in\mathcal{F}}\|g_K\|_{L^2(\omega)}^2\right]^{1/2} \lesssim \mathfrak{F}\|f\|_{L^2(\sigma)}\|\widetilde{g}\|_{L^2(\omega)} \lesssim \mathfrak{F}\|f\|_{L^2(\sigma)}\|g\|_{L^2(\omega)},
\end{aligned}$$

by (2.6) and (4.15).

This completes the proof that the first term on the left side of (4.3) is dominated by the larger term (4.5) on the right side. □

Now we turn to proving that the second term $\mathsf{B}_{mix}(f,g)$ on the left side of (4.3) is dominated by the larger term (4.5) on the right side.

Proof. We claim that the mixed form $\mathsf{B}_{mix}(f,g)$ defined in (4.1) in can be controlled in the same way that the far upper form

$$\begin{aligned}
\mathsf{H}_{far\ upper}(f,g) &= \sum_{\substack{(F,G)\in\mathsf{Far}(\mathcal{F}\times\mathcal{G}) \\ F\subset G}} \left\langle H_\sigma \mathsf{P}_{\mathcal{C}_F^\sigma}^\sigma f, \mathsf{P}_{\mathcal{C}_G}^\omega g\right\rangle_\omega \\
&= \sum_{G\in\mathcal{G}}\sum_{G'\in\mathfrak{C}_{\mathcal{G}}(G)}\sum_{F\in\mathcal{F}:\ F\subset G'}\sum_{I\in\mathcal{C}_F}\sum_{J\in\mathcal{C}_G} \left\langle H_\sigma \triangle_I^\sigma f, \triangle_J^\omega g\right\rangle_\omega \\
&= \sum_{G\in\mathcal{G}}\sum_{G'\in\mathfrak{C}_{\mathcal{G}}(G)}\sum_{F\in\mathcal{F}:\ F\subset G'}\sum_{I\in\mathcal{C}_F}\sum_{J\in\mathcal{C}_G} \left\langle \triangle_I^\sigma f, H_\omega \triangle_J^\omega g\right\rangle_\sigma ,
\end{aligned}$$

would have been controlled in the first part of the proof of Proposition 4.2, where we actually estimated the *dual* form $\mathsf{H}_{far\ lower}$ instead.

Indeed, the only difference between the two forms is that in B_{mix} we have the arrangement of intervals

$$I\subset G'\subset G\subset F, \tag{4.17}$$

with $I\in\mathcal{C}_F^{\sigma}$ and $J\in\mathcal{C}_G^{\omega}$, while in $\mathsf{H}_{far\ upper}$ we have instead the arrangement of intervals

$$I\subset F\subset G'\subset G,$$

with $I\in\mathcal{C}_F^{\sigma}$ and $J\in\mathcal{C}_G^{\omega}$. To control $\mathsf{H}_{far\ upper}$ in the first part of the proof, we would have summed over J to obtain a function $\widehat{g}_F$, and applied the Monotonicity Lemma along with the dual functional energy condition. To control B_{mix} here, we do essentially the same, namely we sum over J to obtain a function $\widehat{g}_I$, and apply the Monotonicity Lemma along with the dual functional energy condition.

We write the mixed form $\mathsf{B}_{mix}(f,g)$ as

$$\begin{aligned}\mathsf{B}_{mix}(f,g) &= \sum_{F\in\mathcal{F}}\mathsf{B}_{mix,F}(f,g);\\ \mathsf{B}_{mix,F}(f,g) &\equiv \sum_{G\sim F}\ \sum_{\substack{(I,J)\in\left(\mathcal{C}_F^{\sigma}\cap\mathcal{C}_{\widetilde{F}}^{\omega}\right)\times\mathcal{C}_G^{\omega}\\ I\Subset G'\in\mathfrak{C}_{\mathcal{G}}(G)}}\left\langle\triangle_I^{\sigma}f,H_{\omega}\triangle_J^{\omega}g\right\rangle_{\sigma}\\ &= \sum_{G\sim F}\ \sum_{G'\in\mathfrak{C}_{\mathcal{G}}(G)}\ \sum_{I\in\mathcal{C}_F:\ I\subset G'}\ \sum_{J\in\mathcal{C}_G^{\omega}}\left\langle\triangle_I^{\sigma}f,H_{\omega}\triangle_J^{\omega}g\right\rangle_{\sigma}\ ,\end{aligned}$$

and prove the estimate,

$$\left|\mathsf{B}_{mix,F}(f,g)\right|\lesssim\left(\mathfrak{NTV}+\mathfrak{F}^{*}\right)\left\|\mathsf{P}_{\mathcal{C}_F^{\sigma}}^{\sigma}f\right\|_{L^2(\sigma)}\left\|\mathsf{P}_{\mathcal{C}_{\widetilde{F}}^{\omega}}^{\omega}g\right\|_{L^2(\omega)}. \tag{4.18}$$

Then we can sum in $F\in\mathcal{F}$ and use Cauchy-Schwarz to obtain

$$\left|\mathsf{B}_{mix}(f,g)\right|\lesssim\left(\mathfrak{NTV}+\mathfrak{F}^{*}\right)\|f\|_{L^2(\sigma)}\|g\|_{L^2(\omega)}.$$

Here are the details.

We have

$$\begin{aligned}\mathsf{B}_{mix,F}(f,g) &= \sum_{G\sim F}\ \sum_{G'\in\mathfrak{C}_{\mathcal{G}}(G)}\ \sum_{I\in\mathcal{C}_F:\ I\subset G'}\ \sum_{J\in\mathcal{C}_G^{\omega}}\left\langle\triangle_I^{\sigma}f,H_{\omega}\triangle_J^{\omega}g\right\rangle_{\sigma}\\ &= \sum_{I\in\mathcal{C}_F}\ \sum_{G\sim F}\ \sum_{\substack{G'\in\mathfrak{C}_{\mathcal{G}}(G)\\ I\subset G'}}\left\langle\triangle_I^{\sigma}f,H_{\omega}\mathsf{P}_{\mathcal{C}_G^{\omega}}^{\omega}g\right\rangle_{\sigma}\\ &= \sum_{I\in\mathcal{C}_F}\left\langle\triangle_I^{\sigma}f,H_{\omega}\widehat{g}_I\right\rangle_{\sigma}\ ,\end{aligned}$$

where

$$\widehat{g}_I\equiv\sum_{G\sim F:}\ \sum_{\substack{G'\in\mathfrak{C}_{\mathcal{G}}(G)\\ I\subset G'}}\ \sum_{J\in\mathcal{C}_G^{\omega}}\triangle_J^{\omega}g,$$

and we recall that $F\in\mathcal{F}$ is fixed. We now decompose this last sum according to whether or not the interval J is disjoint from I: $\widehat{g}_I=g_I^{\natural}+g_I^{\flat}$. As in the earlier argument, the term involving $g_I^\flat$ is handled by Lemma 3.1, and we arrange the intervals J occurring in the sum for

$$g_I^\natural \equiv \sum_{G\sim F} \sum_{\substack{G'\in\mathfrak{C}_{\mathcal{G}}(G)\\ I\subset G'}} \sum_{\substack{J\in\mathcal{C}_G^\omega\\ I\subset J}} \triangle_J^\omega g,$$

into an increasing sequence of consecutive $\mathcal{D}^\omega$-dyadic intervals $\{J_k(I)\}_{k=1}^K$. We also identify the increasing sequence of consecutive $\mathcal{G}$-stopping intervals $\{G_n(I)\}_{n=1}^N$ that contain I, as the subsequence with $\{J_{k_n}(I)\}_{n=1}^N$, i.e. $J_{k_n}(I) = G_n(I)$. It is important to observe that because of the arrangement of intervals in (4.17), we have

$$I\subset G' = G_1(I)\subset J_1(I) = \pi_{\mathcal{D}^\omega} G_1(I)\subset G_N(I)\subset J_K(I) = F.$$

Now we decompose

$$g_I^\natural = g_{I,local}^\natural + g_{I,corona}^\natural + g_{I,stopping}^\natural\ ,$$

where

$$\begin{aligned}
g_{I,local}^\natural &\equiv \left(\mathbb{E}_{G_1(I)}^\omega g - \mathbb{E}_F^\omega g\right)\mathbf{1}_{G_1(I)}\ ,\\
g_{I,corona}^\natural &\equiv \sum_{k\geq k_1:\ \theta(J_k(I))\notin\mathcal{G}} \left(\mathbb{E}_{\theta(J_k(I))}^\omega g - \mathbb{E}_F^\omega g\right)\mathbf{1}_{\theta(J_k(I))}\ ,\\
g_{I,stopping}^\natural &\equiv \sum_{k\geq k_1:\ \theta(J_k(I))\in\mathcal{G}} \left(\mathbb{E}_{\theta(J_k(I))}^\omega g - \mathbb{E}_F^\omega g\right)\mathbf{1}_{\theta(J_k(I))}\ .
\end{aligned}$$

The form corresponding to the local function $g_{I,local}^\natural \equiv g_{G_1(I)}^{@}$ satisfies

$$\begin{aligned}
&\left|\sum_{I\in\mathcal{C}_F^\sigma}\left\langle \triangle_I^\sigma f, H_\omega g_{I,local}^\natural\right\rangle_\sigma\right| = \left|\sum_{G\sim F}\sum_{G'\in\mathfrak{C}_{\mathcal{G}}(G)}\sum_{\substack{I\in\mathcal{C}_F^\sigma\\ G_1(I)=G'}}\left\langle \triangle_I^\sigma f, H_\omega g_G^{@}\right\rangle_\sigma\right|\\
=&\left|\sum_{G\sim F}\left\langle \mathsf{P}_{\mathcal{C}_G^\omega\cap\mathcal{C}_F^\sigma}^\sigma f, H_\omega\left(\left(\mathbb{E}_G^\omega g-\mathbb{E}_F^\omega g\right)\mathbf{1}_G\right)\right\rangle_\sigma\right|\\
\leq&\sum_{G\sim F}\left\|\mathsf{P}_{\mathcal{C}_G^\omega\cap\mathcal{C}_F^\sigma}^\sigma f\right\|_{L^2(\sigma)}\left(\mathbb{E}_G^\omega|g|-\mathbb{E}_F^\omega|g|\right)\sqrt{|G|_\omega}\\
\lesssim&\left(\sum_{G\sim F}\left\|\mathsf{P}_{\mathcal{C}_G\cap\mathcal{C}_F^\sigma}^\sigma f\right\|_{L^2(\sigma)}^2\right)^{\frac12}\left(\sum_{G\sim F}\left(\mathbb{E}_G^\omega|g|\right)^2|G|_\omega\right)^{\frac12}\lesssim\left\|\mathsf{P}_{\mathcal{C}_F^\sigma}^\sigma f\right\|_{L^2(\sigma)}\left\|\mathsf{P}_{\mathcal{C}_{\widetilde{F}}^\omega}^\omega g\right\|_{L^2(\omega)}.
\end{aligned}$$

We next show that the form corresponding to the stopping function $g^{\natural}_{I,stopping}$ is controlled by the $\mathcal{A}_2$ condition alone upon using Lemma 3.16. Indeed, we have

$$\begin{aligned}
&\left|\sum_{I\in\mathcal{C}_F^\sigma}\left\langle \triangle_I^\sigma f, H_\omega g^{\natural}_{I,stopping}\right\rangle_\sigma\right| \\
=&\left|\sum_{I\in\mathcal{C}_F^\sigma}\left\langle \triangle_I^\sigma f, H_\omega\left(\sum_{k\geq k_1:\ \theta(J_k(I))\in\mathcal{G}}\left(\mathbb{E}^\omega_{\theta(J_k(I))}g-\mathbb{E}^\omega_F g\right)\mathbf{1}_{\theta(J_k(I))}\right)\right\rangle_\sigma\right| \\
=&\left|\sum_{G\sim F}\sum_{G'\in\mathfrak{C}_{\mathcal{G}}(G)}\sum_{I\in\mathcal{C}_F^\sigma:\ I\subset\theta G'}\left\langle \triangle_I^\sigma f, H_\omega\left(\left(\mathbb{E}^\omega_{G'}g-\mathbb{E}^\omega_F g\right)\mathbf{1}_{G'}\right)\right\rangle_\sigma\right| \\
\lesssim&\sum_{G\sim F}\sum_{G'\in\mathfrak{C}_{\mathcal{G}}(G)}\sum_{I\in\mathcal{C}_F^\sigma:\ I\subset\theta G'}\left|\left\langle \triangle_I^\sigma f, H_\omega\left(\left(\mathbb{E}^\omega_{G'}|g|\right)\mathbf{1}_{G'}\right)\right\rangle_\sigma\right| \\
\lesssim&\sqrt{\mathcal{A}_2}\left\|\mathsf{P}^\sigma_{\mathcal{C}_F^\sigma}f\right\|_{L^2(\sigma)}\left\|\mathsf{P}^\omega_{\mathcal{C}^\omega_{\widetilde{F}}}g\right\|_{L^2(\omega)},
\end{aligned}$$

by Lemma 3.16. We remark that using $I\subset\theta G'$ in the final sum, we can replace $\sqrt{\mathcal{A}_2}$ with the classical constant $\sqrt{A_2}$ in the estimate above.

Finally, the form corresponding to the corona function $g^{\natural}_{I,corona}$ is controlled by the dual functional energy condition upon applying the Monotonicity Lemma 3.2 as follows. We write

$$\begin{aligned}
g^{\natural}_{I,corona} &= \sum_{k\geq k_1:\ \theta(J_k(I))\notin\mathcal{G}}\left(\mathbb{E}^\omega_{\theta(J_k(I))}g-\mathbb{E}^\omega_F g\right)\mathbf{1}_{\theta(J_k(I))}=\sum_{n=1}^N\beta_{F,G_{n+1}(I)}\mathbf{1}_{G_{n+1}(I)};\\
\beta_{F,G_{n+1}(I)} &\equiv \sum_{k_n\leq k<k_{n+1}:\ \theta(J_k(I))\notin\mathcal{G}}\left(\mathbb{E}^\omega_{\theta(J_k(I))}g-\mathbb{E}^\omega_F g\right)\mathbf{1}_{\theta(J_k(I))},
\end{aligned}$$

where $\beta_{F,G_{n+1}(I)}$ satisfies the pointwise estimate

$$\left|\beta_{F,G_{n+1}(I)}(x)\right|\lesssim\left(\mathbb{E}^\omega_{G_{n+1}(I)}|g|\right)\mathbf{1}_{G_{n+1}(I)\setminus G_n(I)}(x).$$

Thus with

$$\widetilde{f}=\sum_{I\in\mathcal{C}_F^\sigma}\left|\langle f,h_I^\sigma\rangle_\sigma\right|h_I^\sigma,$$

we have

$$\begin{aligned}
&\left|\sum_{I\in\mathcal{C}_F^\sigma}\left\langle \triangle_I^\sigma f, H_\omega g^{\natural}_{I,corona}\right\rangle_\sigma\right| \\
\lesssim&\sum_{I\in\mathcal{C}_F^\sigma}\left\langle \triangle_I^\sigma \widetilde{f}, H_\omega\left(\sum_{n=1}^N\left(\mathbb{E}^\omega_{G_{n+1}(I)}|g|\right)\mathbf{1}_{G_{n+1}(I)\setminus G_n(I)}\right)\right\rangle_\sigma \\
=&\sum_{I\in\mathcal{C}_F^\sigma}\left\langle \triangle_I^\sigma \widetilde{f}, H_\omega\left(\sum_{G\vdash F}\sum_{\substack{G'\in\mathfrak{S}_{\mathcal{G}}(G)\\ I\subset G'}}\sum_{G''\in\mathcal{G}:\ G'\subset G''\subset G}\left(\mathbb{E}^\omega_{\pi_{\mathcal{G}}G''}|g|\right)\mathbf{1}_{\pi_{\mathcal{G}}G''\setminus G''}\right)\right\rangle_\sigma.
\end{aligned}$$

Here the notation $G\vdash F$ means that G is *maximal* with respect to the property $G\sim F$, and $\mathfrak{S}_{\mathcal{G}}(G)$ denotes the successor set of all $G'\in\mathcal{G}$ with $G'\subset G$.

Now we use the arguments surrounding (4.14) and (4.16) in the proof of Proposition 4.2 in order to apply the *dual* of the functional energy condition in (2.6). For convenience we write

$$\sum_{G\vdash F}\sum_{\substack{G'\in\mathfrak{C}_{\mathcal{G}}(G)\\ I\subset G'}}\sum_{G''\in\mathcal{G}:\ G'\subset G''\subset G}\equiv\sum_{G''\bowtie I}.$$

Then with $F(G)\equiv\pi_{\mathcal{F}}G$ and $I\subset I^*\subset\pi_{\mathcal{G}}I$ and $\mathcal{G}_F\equiv\{G\in\mathcal{G}:G\sim F\}$, the error estimate in the the Monotonicity Lemma 3.2 gives

$$\begin{aligned}
(4.19)\quad &\sum_{I\in\mathcal{C}_F^\sigma}\left\langle\triangle_I^\sigma\widetilde{f},H_\omega\left(\sum_{G''\bowtie I}\left(\mathbb{E}_{\pi_{\mathcal{G}}G''}^\omega|g|\right)\mathbf{1}_{\pi_{\mathcal{G}}G''\setminus G''}\right)\right\rangle_\sigma\\
\approx\ &\sum_{I\in\mathcal{C}_F^\sigma}|\langle f,h_I^\sigma\rangle_\sigma|\sum_{G''\bowtie I}\left(\mathbb{E}_{\pi_{\mathcal{G}}G''}^\omega|g|\right)\frac{1}{|I^*|}\mathsf{P}\left(I^*,\mathbf{1}_{\pi_{\mathcal{G}}G''\setminus G''}\omega\right)\ \langle x,h_I^\sigma\rangle_\sigma\\
=\ &\sum_{H\in\mathcal{G}_F}\sum_{I^*\in\mathcal{I}^*(H)}\left\langle\frac{x}{|I^*|},\sum_{\substack{I\in\mathcal{C}_{F(H)}^\sigma\cap\mathcal{C}_H^\omega\\ I\subset I^*}}\triangle_I^\sigma\widetilde{f}\right\rangle_\sigma\mathsf{P}\left(I^*,\sum_{\substack{G''\bowtie I\\ H\subset G''\subset F(H)}}\left(\mathbb{E}_{\pi_{\mathcal{G}}G''}^\omega|g|\right)\mathbf{1}_{\pi_{\mathcal{G}}G''\setminus G''}\omega\right)\\
\leq\ &\sum_{H\in\mathcal{G}_F}\sum_{I^*\in\mathcal{I}^*(H)}\left\langle\frac{x}{|I^*|},\mathbf{1}_{I^*}f_H\right\rangle_\sigma\mathsf{P}\left(I^*,(\mathcal{M}_\omega g)\,\omega\right),
\end{aligned}$$

where the collection of functions

$$f_H\equiv\sum_{I\in\mathcal{C}_{F(H)}^\sigma\cap\mathcal{C}_H^\omega}\triangle_I^\sigma\widetilde{f},\qquad H\in\mathcal{G}_F,$$

is $\mathcal{G}_F$-adapted as in Definition 2.4 above. Indeed, for $I\in\mathcal{C}_{F(H)}^\sigma\cap\mathcal{C}_H^\omega$ we have $\widehat{f_H}(I)=\widehat{\widetilde{f}}(I)\geq 0$, and the orthogonality property

$$\langle f_H,f_{H'}\rangle_\omega=0,\qquad H\neq H'\in\mathcal{G}_F,\tag{4.20}$$

holds since then $\mathcal{C}_H^\omega\cap\mathcal{C}_{H'}^\omega=\emptyset$. Note also that we have the property

$$\left\langle\frac{x}{|I^*|},\mathbf{1}_{I^*}f_H\right\rangle_\sigma=\sum_{\substack{I\in\mathcal{C}_{F(H)}^\sigma\cap\mathcal{C}_H^\omega\\ I\subset I^*}}|\langle f,h_I^\sigma\rangle_\omega|\left\langle\frac{x}{|I^*|},h_I^\sigma\right\rangle_\sigma\geq 0.\tag{4.21}$$

Property (3) of Definition 2.4 holds here with overlap constant $C=1$. Since $\mathcal{G}_F$ is ω-Carleson, we can now apply the inequality *dual* to the functional energy inequality (2.6) to the right side of (4.19), with the choice $h=\mathcal{M}_\omega\mathsf{P}_{\mathcal{C}_{\widetilde{F}}^\omega}^\omega g$. We have the maximal function estimate,

$$\|h\|_{L^2(\omega)}\lesssim\left\|\mathsf{P}_{\mathcal{C}_{\widetilde{F}}^\omega}^\omega g\right\|_{L^2(\omega)},$$

and so altogether we obtain that the right hand side of (4.19) satisfies

$$\begin{aligned}
&\sum_{H\in\mathcal{G}}\sum_{I^*\in\mathcal{I}^*(H)}\left\langle\frac{x}{|I^*|},\mathbf{1}_{I^*}f_H\right\rangle_\sigma \mathsf{P}\left(I^*,\left(\mathcal{M}_\omega g\right)\omega\right)\\
\leq\ &\mathfrak{F}^*\|h\|_\sigma\left[\sum_{H\in\mathcal{G}}\|f_H\|^2_{L^2(\sigma)}\right]^{1/2}\lesssim\mathfrak{F}^*\|\mathsf{P}^\omega_{\mathcal{C}^\omega_{\widetilde{F}}}g\|_{L^2(\omega)}\|\widetilde{f}\|_{L^2(\sigma)}\\
\lesssim\ &\mathfrak{F}^*\left\|\mathsf{P}^\sigma_{\mathcal{C}^\sigma_F}f\right\|_{L^2(\sigma)}\left\|\mathsf{P}^\omega_{\mathcal{C}^\omega_{\widetilde{F}}}g\right\|_{L^2(\omega)},
\end{aligned}$$

by the inequality dual to (2.6), and (4.20). This proves (4.18), and hence completes the proof of Proposition 4.2, but with the larger term (4.5) on the right side of (4.3). □

4.2. **Controlling functional energy.** In this subsection we prove that the functional energy conditions are implied by the strong $\mathcal{A}_2$ and interval testing conditions, thus completing the proof of the Intertwining Proposition 4.2.

Proposition 4.22. *$\mathfrak{F}\lesssim\mathcal{A}_2+\mathfrak{T}$ and $\mathfrak{F}^*\lesssim\mathcal{A}_2+\mathfrak{T}^*$.*

To prove this proposition we fix $\mathcal{F}$ as in (2.6) and set

$$\mu\equiv\sum_{F\in\mathcal{F}}\sum_{J^*\in\mathcal{J}^*(F)}\left\|\mathsf{P}^\omega_{F,J^*}\frac{x}{|J^*|}\right\|^2_{L^2(\omega)}\cdot\delta_{(c(J^*),|J^*|)}\ , \tag{4.23}$$

where $\mathcal{J}^*(F)$ is defined in Definition 2.5, and the projections $\mathsf{P}^\omega_{F,J^*}$ onto Haar functions are defined by

$$\mathsf{P}^\omega_{F,J^*}\equiv\sum_{J\subset J^*:\ J\in\mathcal{J}(F)}\triangle^\omega_J.$$

We can replace x by $x-c$ for any choice of c we wish; the projection is unchanged. Here δ_q denotes a Dirac unit mass at a point q in the upper half plane $\mathbb{R}^2_+$.

We prove the two-weight inequality

$$\|\mathbb{P}(f\sigma)\|_{L^2(\mathbb{R}^2_+,\mu)}\lesssim\|f\|_{L^2(\sigma)}\ , \tag{4.24}$$

for all nonnegative f in $L^2(\sigma)$, noting that $\mathcal{F}$ and f are *not* related here. Above, $\mathbb{P}(\cdot)$ denotes the Poisson extension to the upper half-plane, so that in particular

$$\|\mathbb{P}(f\sigma)\|^2_{L^2(\mathbb{R}^2_+,\mu)}=\sum_{F\in\mathcal{F}}\sum_{J^*\in\mathcal{J}^*(F)}\mathbb{P}(f\sigma)(c(J^*),|J^*|)^2\left\|\mathsf{P}^\omega_{F,J^*}\frac{x}{|J^*|}\right\|^2_{L^2(\omega)}\ ,$$

and so (4.24) implies (2.6) by the Cauchy-Schwarz inequality. By the two-weight inequality for the Poisson operator in [Saw3], inequality (4.24) requires checking these two inequalities

$$\int_{\mathbb{R}^2_+}\mathbb{P}(\mathbf{1}_I\sigma)(x,t)^2\,d\mu(x,t)\equiv\|\mathbb{P}(\mathbf{1}_I\sigma)\|^2_{L^2(\widehat{I},\mu)}\lesssim\left(A_2+\mathfrak{T}^2\right)\sigma(I)\,, \tag{4.25}$$

$$\int_{\mathbb{R}}[\mathbb{P}^*(t\mathbf{1}_{\widehat{I}}\mu)]^2\sigma(dx)\lesssim\mathcal{A}_2\int_{\widehat{I}}t^2\mu(dx,dt), \tag{4.26}$$

for all *dyadic* intervals $I \in \mathcal{D}$, where $\widehat{I} = I \times [0,|I|]$ is the box over I in the upper half-plane, and

$$\mathbb{P}^*(t\mathbf{1}_{\widehat{I}}\mu) = \int_{\widehat{I}} \frac{t^2}{t^2+|x-y|^2}\mu(dy,dt)\,.$$

It is important to note that we can choose for $\mathcal{D}$ any fixed dyadic grid, the compensating point being that the integrations on the left sides of (4.25) and (4.26) are taken over the entire spaces $\mathbb{R}^2_+$ and $\mathbb{R}$ respectively.

Remark 4.27. *There is a gap in the proof of the Poisson inequality at the top of page 542 in* [Saw3]*. However, this gap can be fixed as in* [SaWh] *or* [LaSaUr1]*.*

4.2.1. *The Poisson testing inequality.* We choose the dyadic grid $\mathcal{D}$ in the testing conditions (4.25) and (4.26) to be the grid $\mathcal{D}^\omega$ that arises in the definition of the measure μ in (4.23). In particular all of the intervals J^* lie in the good subgrid $\mathcal{D}^\omega_{good}$ of $\mathcal{D}$. Fix $I \in \mathcal{D}$. We split the integration on the left side of (4.25) into a local and global piece:

$$\int_{\mathbb{R}^2_+} \mathbb{P}\left(\mathbf{1}_I\sigma\right)^2 d\mu = \int_{\widehat{I}} \mathbb{P}\left(\mathbf{1}_I\sigma\right)^2 d\mu + \int_{\mathbb{R}^2_+\setminus\widehat{I}} \mathbb{P}\left(\mathbf{1}_I\sigma\right)^2 d\mu.$$

The global piece turns out to be controlled solely by the $\mathcal{A}_2$ condition, so we leave that term for later, and turn now to estimating the local term.

An important consequence of the fact that I and J^* lie in the same grid $\mathcal{D} = \mathcal{D}^\omega$, is that $\left(c\left(J^*\right),|J^*|\right) \in \widehat{I}$ if and only if $J^* \subset I$. Thus we have

$$\begin{aligned}
&\int_{\widehat{I}} \mathbb{P}\left(\mathbf{1}_I\sigma\right)(x,t)^2\, d\mu\left(x,t\right) \\
&= \sum_{F\in\mathcal{F}} \sum_{J^*\in\mathcal{J}^*(F):\ J^*\subset I} \mathbb{P}\left(\mathbf{1}_I\sigma\right)\left(c\left(J^*\right),|J^*|\right)^2 \left\|\mathsf{P}^\omega_{F,J^*}\frac{x}{|J^*|}\right\|^2_{L^2(\omega)} \\
&= \sum_{F\in\mathcal{F}} \sum_{J^*\in\mathcal{J}^*(F):\ J^*\subset I} \mathsf{P}\left(J^*,\mathbf{1}_I\sigma\right)^2 \|\mathsf{P}^\omega_{F,J^*}\frac{x}{|J^*|}\|^2_{L^2(\omega)}.
\end{aligned}$$

Note that the collections $\mathcal{J}^*\left(F\right)$ are pairwise disjoint for $F \in \mathcal{F}$, and that for $J^* \in \mathcal{J}^*\left(F\right)$ we have

$$\begin{aligned}
\left\|\mathsf{P}^\omega_{F,J^*}\frac{x}{|J^*|}\right\|^2_{L^2(\omega)} &= \left\|\mathsf{P}^\omega_{F,J^*}\left(\frac{x-E^\omega_{J^*}x}{|J^*|}\right)\right\|^2_{L^2(\omega)} \\
&\le \int_{J^*}\left|\frac{x-E^\omega_{J^*}x}{|J^*|}\right|^2 d\omega\left(x\right) = \mathsf{E}\left(J^*,\omega\right)^2 |J^*|_\omega\ .
\end{aligned} \tag{4.28}$$

In the first stage of the proof, we 'create some holes' by restricting the support of σ to the interval I in the sum below.

$$\begin{aligned}
&\sum_{F\in\mathcal{F}} \sum_{J^*\in\mathcal{J}^*(F):\ J^*\subset I} \mathsf{P}\left(J^*,\mathbf{1}_{F\cap I}\sigma\right)^2 \|\mathsf{P}^\omega_{F,J^*}\frac{x}{|J^*|}\|^2_{L^2(\omega)} \\
&= \left\{\sum_{F\in\mathcal{F}:\ F\subset I} + \sum_{F\in\mathcal{F}:\ F\supset I}\right\} \sum_{J^*\in\mathcal{J}^*(F):\ J^*\subset I} \mathsf{P}\left(J^*,\mathbf{1}_{F\cap I}\sigma\right)^2 \|\mathsf{P}^\omega_{F,J^*}\frac{x}{|J^*|}\|^2_{L^2(\omega)} \\
&= A+B.
\end{aligned}$$

Then

$$\begin{aligned} A &\leq \sum_{F\in\mathcal{F}}\sum_{J^*\in\mathcal{J}^*(F):\ J^*\subset I}\mathsf{P}(J^*,\mathbf{1}_{F\cap I}\sigma)^2\mathsf{E}\left(J^*,\omega\right)^2\left|J^*\right|_{\omega} \\ &\leq \mathcal{E}_2\sum_{F\in\mathcal{F}}\sigma(F\cap I)\lesssim\left(A_2+\mathfrak{T}^2\right)\sigma(I)\,. \end{aligned}$$

Here we have used that the constant $\mathcal{E}_2$, defined in the *energy condition*

$$\sup_{\dot\cup_{i=1}^{\infty}I_i\subset I}\sum_i\mathsf{P}(I_i,\mathbf{1}_I\sigma)^2\mathsf{E}\left(I_i,\omega\right)^2\left|I_i\right|_{\omega}\leq\mathcal{E}_2\left|I\right|_{\sigma},\qquad I\in\mathcal{D}^{\sigma},$$

is controlled by the A_2 and testing constant $\mathfrak{T}$ (see [LaSaUr]). We also used that the stopping intervals $\mathcal{F}$ satisfy a σ-Carleson measure estimate,

$$\sum_{F\in\mathcal{F}:\ F\subset F_0}\left|F\right|_{\sigma}\lesssim\left|F_0\right|_{\sigma},$$

which implies that if $\{F_j\}$ are the maximal $F\in\mathcal{F}$ that are contained in I, then

$$\sum_{F\in\mathcal{F}}\sigma(F\cap I)\leq\sum_j\sum_{F\subset F_j}\sigma(F)\lesssim\sum_j\sigma(F_j)\leq\sigma(I).$$

Now let $\widetilde{\mathcal{J}}(I)$ consist of those $J^*\subset I$ that lie in $\mathcal{J}^*(F)$ for some $F\supset I$. For $J^*\in\widetilde{\mathcal{J}}(I)$ there are only two possibilities:

$$J^*\Subset I\text{ or }J^*\not\Subset I.$$

If $J^*\Subset I$ and $F\supset I$, then $J^*\Subset F$ by the definition of J^* good, and then by Property (3) in the definition of $\mathcal{J}(F)$, Definition 2.4, it follows that the intervals $J^*\in\widetilde{\mathcal{J}}(I)$ with $J^*\Subset I$ have overlap bounded by C, independent of I. As for the other case $J^*\in\widetilde{\mathcal{J}}(I)$ and $J^*\not\Subset I$, there are at most 2^{r+1} such intervals J^*, and they can be easily estimated without regard to their overlap if we let F_{J^*} be the unique interval $F_{J^*}\supset I$ with $J^*\in\mathcal{J}^*(F_{J^*})$. Inequality (4.28) then shows that term B satisfies

$$\begin{aligned} B &\leq \left\{\sum_{J^*\in\widetilde{\mathcal{J}}(I):\ J^*\Subset I}+\sum_{J^*\in\widetilde{\mathcal{J}}(I):\ J^*\not\Subset I}\right\}\mathsf{P}\left(J^*,\mathbf{1}_I\sigma\right)^2\left\|\mathsf{P}^{\omega}_{F_I,J^*}\frac{x}{|J^*|}\right\|^2_{L^2(\omega)} \\ &\lesssim \sum_{J^*\in\widetilde{\mathcal{J}}(I):\ J^*\Subset I}\mathsf{P}\left(J^*,\mathbf{1}_I\sigma\right)^2\mathsf{E}\left(J^*,\omega\right)^2\left|J^*\right|_{\omega} \\ &\quad +2^{r+1}\sup_{J^*\in\widetilde{\mathcal{J}}(I):\ J^*\not\Subset I}\mathsf{P}\left(J^*,\mathbf{1}_I\sigma\right)^2\left\|\mathsf{P}^{\omega}_{F,J^*}\frac{x}{|J^*|}\right\|^2_{L^2(\omega)} \\ &\leq \mathcal{E}_2\sigma(I)+2^{r+1}\mathcal{E}_2\sigma(I)\lesssim\left(A_2+\mathfrak{T}^2\right)\sigma(I)\,, \end{aligned}$$

since the intervals $J^*\in\widetilde{\mathcal{J}}(I)$ with $J^*\Subset I$ have overlap bounded by C, independent of I.

It remains then to show the following inequality with 'holes':

$$\sum_{F\in\mathcal{F}_I}\sum_{J^*\in\mathcal{J}^*(F)}\mathsf{P}\left(J^*,\mathbf{1}_{I\setminus F}\sigma\right)^2\left\|\mathsf{P}^{\omega}_{F,J^*}\frac{x}{|J^*|}\right\|^2_{L^2(\omega)}\lesssim\left(A_2+\mathfrak{T}^2\right)\sigma(I)\,,\tag{4.29}$$

where $\mathcal{F}_I$ consists of those $F \in \mathcal{F}$ with $F \subset I$. Because of the holes, we are able to express this inequality in dual language via the pointwise control given in the Monotonicity Lemma 3.2:

$$\begin{aligned}\mathbf{E} &\equiv \left\langle H\left(\mathbf{1}_{I\setminus F}\sigma\right), h_J^\omega\right\rangle_\omega - \frac{1}{|J^*|}\mathsf{P}\left(J^*,\sigma\right)\ \langle x, h_J^\omega\rangle \\ &= O\left(\frac{|J|}{|J^*|^2}\widetilde{\mathsf{P}}\left(J^*, \mathbf{1}_{I\setminus F}\sigma\right)\ \langle x, h_J^\omega\rangle\right).\end{aligned} \tag{4.30}$$

We will prove below that for any dyadic interval I,

$$\sum_{F\in\mathcal{F}_I}\sum_{J^*\in\mathcal{J}^*(F)}\left\langle H\left(\mathbf{1}_{I\setminus F}\sigma\right), \widetilde{g}_{F,J^*}\right\rangle_\omega \leq \mathfrak{T}\sigma(I)^{1/2}\left\|\sum_{F\in\mathcal{F}}\sum_{J^*\in\mathcal{J}^*(F)} g_{F,J^*}\right\|_{L^2(\omega)}, \tag{4.31}$$

where the functions

$$g_{F,J^*} = \sum_{J\in\mathcal{J}(F):\ J\subset J^*}\left\langle g_{F,J^*}, h_J^\omega\right\rangle_\omega\ h_J^\omega,$$

satisfy $\mathsf{P}^\omega_{F,J^*} g_{F,J^*} = g_{F,J^*}$, and are pairwise orthogonal in (F, J^*); and where the functions

$$\widetilde{g}_{F,J^*} = \sum_{J\in\mathcal{J}(F):\ J\subset J^*}\left|\left\langle g_{F,J^*}, h_J^\omega\right\rangle_\omega\right|\ h_J^\omega,$$

satisfy the same conditions as the g_{F,J^*} and with the same $L^2(\omega)$ norms. Using the equivalence (3.4) in the Monotonicity lemma, together with $\left\langle \widetilde{g}_{F,J^*}, h_J^\omega\right\rangle_\omega \geq 0$, we get

$$\begin{aligned}&\left|\left\langle H\left(\mathbf{1}_{I\setminus F}\sigma\right), g_{F,J^*}\right\rangle_\omega\right| \\ &= \left|\sum_{J\in\mathcal{J}(F):\ J\subset J^*}\left\langle g_{F,J^*}, h_J^\omega\right\rangle_\omega\ \left\langle H\left(\mathbf{1}_{I\setminus F}\sigma\right), h_J^\omega\right\rangle_\omega\right| \\ &\leq \sum_{J\in\mathcal{J}(F):\ J\subset J^*}\left\langle \widetilde{g}_{F,J^*}, h_J^\omega\right\rangle_\omega\ \left\langle H\left(\mathbf{1}_{I\setminus F}\sigma\right), h_J^\omega\right\rangle_\omega \\ &\approx \sum_{J\in\mathcal{J}(F):\ J\subset J^*}\left\langle \widetilde{g}_{F,J^*}, h_J^\omega\right\rangle_\omega\ \frac{1}{|J^*|}\mathsf{P}\left(J^*, \mathbf{1}_{I\setminus F}\sigma\right)\ \langle x, h_J^\omega\rangle \\ &= \frac{1}{|J^*|}\mathsf{P}\left(J^*, \mathbf{1}_{I\setminus F}\sigma\right)\ \left\langle \mathsf{P}^\omega_{F,J^*}x, \widetilde{g}_{F,J^*}\right\rangle_\omega.\end{aligned} \tag{4.32}$$

Now

$$\sum_{F\in\mathcal{F}_I}\sum_{J^*\in\mathcal{J}^*(F)}\frac{1}{|J^*|}\mathsf{P}\left(J^*, \mathbf{1}_{I\setminus F}\sigma\right)\ \left\langle \mathsf{P}^\omega_{F,J^*}x, g_{F,J^*}\right\rangle_\omega$$

can be viewed as an inner product, and since (4.32) and (4.31) give

$$\begin{aligned}&\left|\sum_{F\in\mathcal{F}_I}\sum_{J^*\in\mathcal{J}^*(F)}\frac{1}{|J^*|}\mathsf{P}\left(J^*, \mathbf{1}_{I\setminus F}\sigma\right)\ \left\langle \mathsf{P}^\omega_{F,J^*}x, g_{F,J^*}\right\rangle_\omega\right| \\ &\leq \left(\mathfrak{T}+\sqrt{A_2}\right)\sigma(I)^{1/2}\left\|\sum_{F\in\mathcal{F}_I}\sum_{J^*\in\mathcal{J}^*(F)} g_{F,J^*}\right\|_{L^2(\omega)},\end{aligned} \tag{4.33}$$

since $\|\widetilde{g}_{F,J^*}\|_{L^2(\omega)} = \|g_{F,J^*}\|_{L^2(\omega)}$, it then follows by duality that

$$\sum_{F\in\mathcal{F}_I}\sum_{J^*\in\mathcal{J}^*(F)} \frac{1}{|J^*|^2}\mathsf{P}\left(J^*,\mathbf{1}_{I\setminus F}\sigma\right)^2 \left\|\mathsf{P}^{\omega}_{F,J^*}x\right\|^2_{L^2(\omega)} \lesssim \left(\mathfrak{T}^2+A_2\right)\sigma(I),$$

which is (4.29). Thus it remains to prove (4.31).

The key to this is to note that we can now 'plug the hole' we created above in order to dualize (4.29) via the Monotonicity Lemma. We have

$$\begin{aligned}\sum_{F\in\mathcal{F}_I}\sum_{J^*\in\mathcal{J}^*(F)} \left\langle H\left(\mathbf{1}_F\sigma\right),\widetilde{g}_{F,J^*}\right\rangle_\omega &\le \mathfrak{T}\sum_{F\in\mathcal{F}_I}\sigma(F\cap I)^{1/2}\left\|\sum_{J^*\in\mathcal{J}^*(F)}\widetilde{g}_{F,J^*}\right\|_{L^2(\omega)}\\ &\le \mathfrak{T}\left[\sum_{F\in\mathcal{F}_I}\sigma(F\cap I)\times\sum_{F\in\mathcal{F}_I}\left\|\sum_{J^*\in\mathcal{J}^*(F)}\widetilde{g}_{F,J^*}\right\|^2_{L^2(\omega)}\right]^{1/2}.\end{aligned}$$

And this gives us the inequality we want. The functions $\widetilde{g}_{F,J^*}$ are pairwise orthogonal in $L^2(w)$. And the intervals $\mathcal{F}$ are stopping intervals, hence satisfy a σ-Carleson measure estimate, which if $\{F_j\}$ are the maximal such F contained in I leads to

$$\sum_{F\in\mathcal{F}_I}\sigma(F\cap I)\le\sum_j\sum_{F\subset F_j}\sigma(F)\lesssim\sum_j\sigma(F_j)\le\sigma(I).$$

But it is trivial that

$$\sum_{F\in\mathcal{F}_I}\sum_{J^*\in\mathcal{J}^*(F)}\left\langle H\left(\mathbf{1}_I\sigma\right),\widetilde{g}_{F,J^*}\right\rangle_\omega \lesssim \mathfrak{T}\sigma(I)^{1/2}\left\|\sum_{F\in\mathcal{F}}\sum_{J\in\mathcal{J}(F)}\widetilde{g}_{F,J^*}\right\|_{L^2(\omega)},$$

and combined with the previous display and $H\left(\mathbf{1}_{I\setminus F}\sigma\right)=H\left(\mathbf{1}_I\sigma\right)-H\left(\mathbf{1}_F\sigma\right)$, this yields (4.31). This completes the proof of the local part of the first testing condition (4.25).

Now we turn to proving the following estimate for the global part of the first testing condition (4.25):

$$\int_{\mathbb{R}^2_+\setminus\widehat{I}}\mathbb{P}\left(\mathbf{1}_I\sigma\right)^2 d\mu\lesssim\mathcal{A}_2\left|I\right|_\sigma.$$

We begin by decomposing the integral on the left into four pieces:

$$\begin{aligned}\int_{\mathbb{R}^2_+\setminus\widehat{I}}\mathbb{P}\left(\mathbf{1}_I\sigma\right)^2 d\mu &= \sum_{J^*:\ (c(J^*),|J^*|)\in\mathbb{R}^2_+\setminus\widehat{I}}\mathbb{P}\left(\mathbf{1}_I\sigma\right)\left(c\left(J^*\right),\left|J^*\right|\right)^2\sum_{F\sim J^*}\left\|\mathsf{P}^{\omega}_{F,J^*}\frac{x}{|J^*|}\right\|^2_{L^2(\omega)}\\ &= \left\{\sum_{\substack{J^*\cap 3I=\emptyset\\ |J^*|\le|I|}}+\sum_{J^*\subset 3I\setminus I}+\sum_{\substack{J^*\cap I=\emptyset\\ |J^*|>|I|}}+\sum_{J^*\supsetneqq I}\right\}\mathbb{P}\left(\mathbf{1}_I\sigma\right)\left(c\left(J^*\right),\left|J^*\right|\right)^2\sum_{F\sim J^*}\left\|\mathsf{P}^{\omega}_{F,J^*}\frac{x}{|J^*|}\right\|^2_{L^2(\omega)}\\ &= A+B+C+D.\end{aligned}$$

We further decompose term A according to the length of J^* and its distance from I, and then use (4.28) and $\mathsf{E}(J^*,\omega)\leq 1$ to obtain:

$$\begin{aligned} A &\lesssim \sum_{n=0}^{\infty}\sum_{k=1}^{\infty}\sum_{\substack{J^*\subset 3^{k+1}I\setminus 3^kI=\emptyset\\ |J^*|=2^{-n}|I|}}\left(\frac{2^{-n}|I|}{dist(J^*,I)^2}|I|_\sigma\right)^2|J^*|_\omega \\ &\lesssim \sum_{n=0}^{\infty}2^{-2n}\sum_{k=1}^{\infty}\frac{|I|^2|I|_\sigma\left|3^{k+1}I\setminus 3^kI\right|_\omega}{|3^kI|^4}|I|_\sigma \\ &\lesssim \sum_{n=0}^{\infty}2^{-2n}\sum_{k=1}^{\infty}3^{-2k}\left\{\frac{\left|3^{k+1}I\right|_\sigma\left|3^{k+1}I\right|_\omega}{|3^kI|^2}\right\}|I|_\sigma\lesssim A_2|I|_\sigma\,. \end{aligned}$$

We further decompose term B according to the length of J^* and then use the Poisson inequality (3.15),

$$\mathsf{P}(J^*,\mathbf{1}_I\sigma)^2\lesssim\left(\frac{|J^*|}{|I|}\right)^{2-4\varepsilon}\mathsf{P}(I,\mathbf{1}_I\sigma)^2\,,$$

in Lemma 3.14, which requires the fact that our grid $\mathcal{D}^\omega_{good}$ is a good subgrid of $\mathcal{D}=\mathcal{D}^\omega$ as defined in Subsection 3.3. We then obtain

$$\begin{aligned} B &\lesssim \sum_{n=0}^{\infty}\sum_{\substack{J^*\subset 3I\setminus I\\ |J^*|=2^{-n}|I|}}\left(2^{-n}\right)^{2-4\varepsilon}\left(\frac{|I|_\sigma}{|I|}\right)^2|J^*|_\omega \\ &\leq \sum_{n=0}^{\infty}\left(2^{-n}\right)^{2-4\varepsilon}\frac{|3I|_\sigma|3I|_\omega}{|3I|}|I|_\sigma\lesssim A_2|I|_\sigma\,. \end{aligned}$$

For term C we will have to group the intervals J^* into blocks B_i, and then exploit the mutual orthogonality in the pairs (F,J^*) of the projections $\mathsf{P}^\omega_{F,J^*}$ defining μ, in order to avoid overlapping estimates. We first split the sum according to whether or not I intersects the triple of J^*:

$$\begin{aligned} C &\approx \left\{\sum_{\substack{J^*:\ I\cap 3J^*=\emptyset\\ |J^*|>|I|}}+\sum_{\substack{J^*:\ I\subset 3J^*\setminus J^*\\ |J^*|>|I|}}\right\}\left(\frac{|J^*|}{dist(J^*,I)^2}|I|_\sigma\right)^2\sum_{F\sim J^*}\left\|\mathsf{P}^\omega_{F,J^*}\frac{x}{|J^*|}\right\|^2_{L^2(\omega)} \\ &= C_1+C_2. \end{aligned}$$

Let $\{B_i\}_{i=1}^{\infty}$ be the maximal intervals in the collection of triples

$$\{3J^*:|J^*|>|I| \text{ and } 3J^*\cap I=\emptyset\}\,,$$

arranged in order of increasing side length. Below we will use the simple fact that the intervals B_i have bounded overlap, $\sum_{i=1}^{\infty}\mathbf{1}_{B_i}\leq 3$. Now we further decompose the sum in C_1 by grouping the intervals J^* into the blocks B_i, and then using that

$\mathsf{P}^{\omega}_{F,J^*}x = \mathsf{P}^{\omega}_{F,J^*}(x - c(B_i))$ along with the mutual orthogonality of the $\mathsf{P}^{\omega}_{F,J^*}$:

$$\begin{aligned}
C_1 &\leq \sum_{i=1}^{\infty} \sum_{J^*:\ 3J^* \subset B_i} \left(\frac{|J^*|}{dist(J^*,I)^2} |I|_\sigma \right)^2 \sum_{F \sim J^*} \left\| \mathsf{P}^{\omega}_{F,J^*} \frac{x}{|J^*|} \right\|^2_{L^2(\omega)} \\
&\lesssim \sum_{i=1}^{\infty} \left(\frac{1}{dist(B_i,I)^2} |I|_\sigma \right)^2 \sum_{J^*:\ 3J^* \subset B_i} \sum_{F \sim J^*} \left\| \mathsf{P}^{\omega}_{F,J^*} x \right\|^2_{L^2(\omega)} \\
&\lesssim \sum_{i=1}^{\infty} \left(\frac{1}{dist(B_i,I)^2} |I|_\sigma \right)^2 \left\| \mathbf{1}_{B_i}(x - c(B_i)) \right\|^2_{L^2(\omega)} \\
&\lesssim \sum_{i=1}^{\infty} \left(\frac{1}{dist(B_i,I)^2} |I|_\sigma \right)^2 |B_i|^2 |B_i|_\omega \\
&\lesssim \left\{ \sum_{i=1}^{\infty} \frac{|B_i|_\omega |I|_\sigma}{|B_i|^2} \right\} |I|_\sigma \lesssim \mathcal{A}_2 |I|_\sigma
\end{aligned}$$

since $dist(B_i,I) \approx |B_i|$ and

$$\begin{aligned}
\sum_{i=1}^{\infty} \frac{|B_i|_\omega |I|_\sigma}{|B_i|^2} &= \frac{|I|_\sigma}{|I|} \sum_{i=1}^{\infty} \frac{|I|}{|B_i|^2} |B_i|_\omega \\
&\approx \frac{|I|_\sigma}{|I|} \sum_{i=1}^{\infty} \int_{B_i} \frac{|I|}{dist(x,I)^2} d\omega(x) \\
&\lesssim \frac{|I|_\sigma}{|I|} \mathsf{P}(I,\omega) \leq \mathcal{A}_2,
\end{aligned}$$

since $\sum_{i=1}^{\infty} \mathbf{1}_{B_i} \leq 3$.

Next we turn to estimating term C_2 where the triple of J^* contains I but J^* itself does not. Note that there are at most two such intervals J^* of a given length, one to the left and one to the right of I. So with this in mind we sum over the intervals J^* according to their lengths and use (4.28) to obtain

$$\begin{aligned}
C_2 &= \sum_{n=0}^{\infty} \sum_{\substack{J^*:\ I \subset 3J^* \setminus J^* \\ |J^*| = 2^n |I|}} \left(\frac{|J^*|}{dist(J^*,I)^2} |I|_\sigma \right)^2 \sum_{F \sim J^*} \left\| \mathsf{P}^{\omega}_{F,J^*} \frac{x}{|J^*|} \right\|^2_{L^2(\omega)} \\
&\lesssim \sum_{n=0}^{\infty} \left(\frac{|I|_\sigma}{|2^n I|} \right)^2 |3 \cdot 2^n I|_\omega = \left\{ \frac{|I|_\sigma}{|I|} \sum_{n=0}^{\infty} \frac{|3 \cdot 2^n I|_\omega}{|2^n I|^2} \right\} |I|_\sigma \\
&\lesssim \left\{ \frac{|I|_\sigma}{|I|} \mathsf{P}(I,\omega) \right\} |I|_\sigma \leq \mathcal{A}_2 |I|_\sigma ,
\end{aligned}$$

since

$$\sum_{n=0}^{\infty} \frac{|3 \cdot 2^n I|_\omega}{|2^n I|^2} = \int \sum_{n=0}^{\infty} \frac{1}{|2^n I|^2} \mathbf{1}_{3 \cdot 2^n I}(x)\ d\omega(x) \lesssim \mathsf{P}(I,\omega).$$

Finally, we turn to term D, which is handled in the same way as term C_2. The intervals J^* occurring here are included in the set of ancestors $A_k \equiv \pi^{(k)}_{\mathcal{D}} I$ of I,

$1 \leq k < \infty$. We thus have

$$\begin{aligned} D &= \sum_{k=1}^{\infty} \mathbb{P}\left(\mathbf{1}_I \sigma\right)\left(c\left(A_k\right), \left|A_k\right|\right)^2 \sum_{F \sim A_k} \left\| \mathsf{P}^{\omega}_{F,J^*} \frac{x}{\left|A_k\right|} \right\|^2_{L^2(\omega)} \\ &\lesssim \sum_{k=1}^{\infty} \left(\frac{\left|I\right|_\sigma}{\left|A_k\right|} \right)^2 \left|A_k\right|_\omega = \left\{ \frac{\left|I\right|_\sigma}{\left|I\right|} \sum_{k=1}^{\infty} \frac{\left|I\right|}{\left|A_k\right|^2} \left|A_k\right|_\omega \right\} \left|I\right|_\sigma \\ &\lesssim \left\{ \frac{\left|I\right|_\sigma}{\left|I\right|} \mathsf{P}\left(I, \omega\right) \right\} \left|I\right|_\sigma \lesssim \mathcal{A}_2 \left|I\right|_\sigma , \end{aligned}$$

since

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{\left|I\right|}{\left|A_k\right|^2} \left|A_k\right|_\omega &= \int \sum_{k=1}^{\infty} \frac{\left|I\right|}{\left|A_k\right|^2} \mathbf{1}_{A_k(x)} d\omega\left(x\right) \\ &\lesssim \int \frac{\left|I\right|}{\left|I\right|^2 + dist\left(x, I\right)^2} d\omega\left(x\right) = \mathsf{P}\left(I, \omega\right). \end{aligned}$$

Remark 4.34. *The reduction to the testing condition here seems to be essential as one can't 'plug the hole' in the function setting.*

4.2.2. *The dual Poisson testing inequality.* Again we split the integration on the left side of (4.26) into local and global parts:

$$\int_{\mathbb{R}} [\mathbb{P}^*(t\mathbf{1}_{\widehat{I}}\mu)]^2 \sigma = \int_I [\mathbb{P}^*(t\mathbf{1}_{\widehat{I}}\mu)]^2 \sigma + \int_{\mathbb{R}\setminus I} [\mathbb{P}^*(t\mathbf{1}_{\widehat{I}}\mu)]^2 \sigma.$$

We begin with the local part. Note that

$$\int_{\widehat{I}} t^2 d\mu\left(x, t\right) = \sum_{F \in \mathcal{F}} \sum_{\substack{J^* \in \mathcal{J}^*(F) \\ J^* \subset I}} \|\mathsf{P}^{\omega}_{F,J^*} z\|^2_{L^2(\omega)} , \tag{4.35}$$

where we are using the dummy variable z to denote the argument of $\mathsf{P}^{\omega}_{F,J^*}$ so as to avoid confusion with the integration variable x in $d\sigma\left(x\right)$. Compute

$$\mathbb{P}^* \left(t\mathbf{1}_{\widehat{I}}\mu\right)\left(x\right) = \sum_{F \in \mathcal{F}} \sum_{\substack{J^* \in \mathcal{J}^*(F) \\ J^* \subset I}} \frac{\|\mathsf{P}^{\omega}_{F,J^*} z\|^2_{L^2(\omega)}}{|J^*|^2 + |x - c(J^*)|^2} \tag{4.36}$$

And so, it makes sense to expand the square. The diagonal term is

$$\sum_{F \in \mathcal{F}} \sum_{\substack{J^* \in \mathcal{J}^*(F) \\ J^* \subset I}} \int \left[\frac{\|\mathsf{P}^{\omega}_{F,J^*} z\|^2_{L^2(\omega)}}{|J|^2 + |x - c(J)|^2} \right]^2 d\sigma(x) \leq M_1 \cdot \sum_{F \in \mathcal{F}} \sum_{\substack{J^* \in \mathcal{J}^*(F) \\ J^* \subset I}} \|\mathsf{P}^{\omega}_{F,J^*} z\|^2_{L^2(\omega)} \tag{4.37}$$

$$\textit{where} \quad M_1 \equiv \sup_{F \in \mathcal{F}} \sup_{\substack{J^* \in \mathcal{J}^*(F) \\ J^* \subset I}} \int \frac{\|\mathsf{P}^{\omega}_{F,J^*} z\|^2_{L^2(\omega)}}{(|J^*|^2 + |x - c(J^*)|^2)^2} \sigma(dx). \tag{4.38}$$

But, by inspection, M_1 is dominated by the $\mathcal{A}_2$ constant. Indeed, for any J^*, we have by (4.28)

$$\int \frac{\|\mathsf{P}^{\omega}_{F,J^*} z\|^2_{L^2(\omega)}}{(|J^*|^2 + |x - c(J^*)|^2)^2} \sigma(dx) \leq \frac{|J^*|_\omega}{|J^*|} \int \frac{|J^*|^3}{(|J^*|^2 + |x - c(J^*)|^2)^2} \sigma(dx) \leq \mathcal{A}_2 .$$

Having fixed ideas, we fix an integer s, and consider those intervals J and J' with $|J'| = 2^{-s}|J|$, where we are now dropping the superscripts $*$ from the intervals J^*, but *not* from $\mathcal{J}^*(F)$, for clarity of display. The expression to control is

$$\begin{aligned} T_s &\equiv \sum_{F\in\mathcal{F}} \sum_{J\in\mathcal{J}(F)} \sum_{F'\in\mathcal{F}} \sum_{\substack{J'\in\mathcal{J}(F) \\ |J'|=2^{-s}|J|}} \int_I \frac{\|\mathsf{P}^{\omega}_{F,J^*}z\|^2_{L^2(\omega)}}{|J|^2+|x-c(J)|^2} \frac{\|\mathsf{P}^{\omega}_{F',J'}z\|^2_{L^2(\omega)}}{|J'|^2+|x-c(J')|^2}\sigma(dx) \\ &\leq M_2 \sum_{F\in\mathcal{F}} \sum_{J\in\mathcal{J}(F)} \|\mathsf{P}^{\omega}_{F,J^*}z\|^2_{\omega} \end{aligned}$$

$$\textit{where} \quad M_2 \equiv \sup_{F\in\mathcal{F}} \sup_{J\in\mathcal{J}(F)} \sum_{F'\in\mathcal{F}} \sum_{\substack{J'\in\mathcal{J}(F) \\ |J'|=2^{-s}|J|}} \int_I \frac{1}{|J|^2+|x-c(J)|^2} \frac{\|\mathsf{P}^{\omega}_{F',J'}z\|^2_{L^2(\omega)}}{|J'|^2+|x-c(J')|^2}\sigma(dx).$$

We claim the term M_2 is at most a constant times $\mathcal{A}_2 2^{-s}$. To see , fix J as in the definition of M_2, and use (4.28) to estimate the integral on the right by

$$\frac{|J'|_\omega}{|J'|} \int_I \frac{|J'|^2}{|J|^2+|x-c(J)|^2} \frac{|J'|}{|J'|^2+|x-c(J')|^2}\sigma(dx) \lesssim \mathcal{A}_2 \frac{2^{-2s}}{1+n^2}$$

where n is an integer chosen so that $(n-1)|J| \leq dist(J,J') \leq n|J|$. Then estimate the sum over J' as follows.

$$\sum_{F'\in\mathcal{F}} \sum_{\substack{J'\in\mathcal{J}^*(F')\,:\,|J'|=2^{-s}|J| \\ (n-1)|J|\leq dist(J,J')\leq n|J|}} \frac{2^{-2s}}{1+n^2} \lesssim \frac{2^{-s}}{1+n^2}\,.$$

because the relative lengths of J and J' are fixed. This is summable over $n \in \mathbb{N}$ to 2^{-s}, so it completes our proof of the local part of the second testing condition (4.26).

It remains to prove the following estimate for the global part of the second testing condition (4.26):

$$\int_{\mathbb{R}\setminus I} [\mathbb{P}^*(t\mathbf{1}_{\widehat{I}}\mu)]^2\sigma \lesssim \mathcal{A}_2 \left|I\right|_\sigma.$$

We decompose the integral on the left into two pieces:

$$\int_{\mathbb{R}\setminus I} [\mathbb{P}^*(t\mathbf{1}_{\widehat{I}}\mu)]^2\sigma = \int_{\mathbb{R}\setminus 3I} [\mathbb{P}^*(t\mathbf{1}_{\widehat{I}}\mu)]^2\sigma + \int_{3I\setminus I} [\mathbb{P}^*(t\mathbf{1}_{\widehat{I}}\mu)]^2\sigma = A + B.$$

We further decompose term A in annuli and use (4.36) to obtain

$$\begin{aligned} A &= \sum_{n=1}^{\infty} \int_{3^{n+1}I\setminus 3^n I} [\mathbb{P}^*(t\mathbf{1}_{\widehat{I}}\mu)]^2\sigma \\ &= \sum_{n=1}^{\infty} \int_{3^{n+1}I\setminus 3^n I} \left[\sum_{F\in\mathcal{F}} \sum_{\substack{J^*\in\mathcal{J}^*(F) \\ J^*\subset I}} \frac{\|\mathsf{P}^{\omega}_{F,J^*}z\|^2_{L^2(\omega)}}{|J^*|^2+|x-c(J^*)|^2} \right]^2 d\sigma(x) \\ &\lesssim \sum_{n=1}^{\infty} \int_{3^{n+1}I\setminus 3^n I} \left[\sum_{F\in\mathcal{F}} \sum_{\substack{J^*\in\mathcal{J}^*(F) \\ J^*\subset I}} \|\mathsf{P}^{\omega}_{F,J^*}z\|^2_{L^2(\omega)} \right]^2 \frac{1}{|3^n I|^4} d\sigma(x)\,. \end{aligned}$$

Now use (4.35) and

$$\int_{\widehat{I}} t^2 d\mu = \sum_{F\in\mathcal{F}} \sum_{\substack{J^*\in\mathcal{J}^*(F)\\ J^*\subset I}} \|\mathsf{P}^{\omega}_{F,J^*} z\|^2_{L^2(\omega)} \lesssim \|\mathbf{1}_I (z-c(I))\|^2_{L^2(\omega)} \leq |I|^2 |I|_\omega$$

to obtain that

$$\begin{aligned} A &\lesssim \sum_{n=1}^{\infty} \int_{3^{n+1}I\setminus 3^n I} \left[\int_{\widehat{I}} t^2 d\mu\right] \left[|I|^2 |I|_\omega\right] \frac{1}{|3^n I|^4} d\sigma(x) \\ &\lesssim \left\{\sum_{n=1}^{\infty} 3^{-2n} \frac{|3^{n+1}I|_\omega |3^{n+1}I|_\sigma}{|3^{n+1}I|^2}\right\} \left[\int_{\widehat{I}} t^2 d\mu\right] \lesssim A_2 \int_{\widehat{I}} t^2 d\mu. \end{aligned}$$

Finally, we estimate term B by using (4.36) to write

$$B = \int_{3I\setminus I} \left[\sum_{F\in\mathcal{F}} \sum_{\substack{J^*\in\mathcal{J}^*(F)\\ J^*\subset I}} \frac{\|\mathsf{P}^{\omega}_{F,J^*} z\|^2_{L^2(\omega)}}{|J^*|^2 + |x-c(J^*)|^2}\right]^2 d\sigma(x),$$

and then expanding the square and calculating as in the proof of the local part given earlier. The details are similar and left to the reader.

4.3. **General stopping data.** Here we prove Proposition 4.4. Let $f\in L^2(\sigma)$ and $g\in L^2(\omega)$. Suppose we are given stopping data for f as in Definition 1.8, i.e. a positive constant $C_0\geq 4$, a subset $\mathcal{F}$ of the dyadic grid $\mathcal{D}^\sigma$, and a corresponding sequence $\alpha_{\mathcal{F}} \equiv \{\alpha_{\mathcal{F}}(F)\}_{F\in\mathcal{F}}$ of nonnegative numbers $\alpha_{\mathcal{F}}(F)\geq 0$ satisfying

(1) $\mathbb{E}^\sigma_I |f| \leq \alpha_{\mathcal{F}}(F)$ for all $I\in\mathcal{C}_F$ and $F\in\mathcal{F}$,
(2) $\sum_{F'\preceq F} |F'|_\sigma \leq C_0 |F|_\sigma$ for all $F\in\mathcal{F}$,
(3) $\sum_{F\in\mathcal{F}} \alpha_{\mathcal{F}}(F)^2 |F|_\sigma \leq \left\|\sum_{F\in\mathcal{F}} \alpha_{\mathcal{F}}(F)\mathbf{1}_F\right\|^2_{L^2(\sigma)} \leq C_0^2 \|f\|^2_{L^2(\sigma)}$,
(4) $\alpha_{\mathcal{F}}(F)\leq \alpha_{\mathcal{F}}(F')$ whenever $F',F\in\mathcal{F}$ with $F'\subset F$.

Note that we have here included in property (3) the quasiorthogonality inequality (1.10). Similarly, let $\mathcal{G}$ and $\beta_{\mathcal{G}} \equiv \left\{\beta_{\mathcal{G}}(G)\right\}_{G\in\mathcal{G}}$ be stopping data for g. We begin by following the proof of Proposition 4.2, which makes no use of the explicit form of stopping data until we get to the telescoping sums in inequality (4.9), which now becomes

$$\left|f^\natural_G(x)\right| = \left|\sum_{\ell=k}^{\infty} \triangle^\sigma_{I_\ell} f(x)\right| = \left|\mathbb{E}^\sigma_{\theta(I_k(G))} f - \mathbb{E}^\sigma_{I_K} f\right| \lesssim \alpha_{\mathcal{F}}(F_{n+1}) + \alpha_{\mathcal{F}}(K) \leq 2\alpha_{\mathcal{F}}(F_{n+1}),$$

by properties (1) and (4) above. Then we proceed with the decomposition

$$f^\natural_G = f^\natural_{G,local} + f^\natural_{G,corona} + f^\natural_{G,stopping}.$$

The estimates

$$\left|\sum_{G\in\mathcal{G}}\int H_{\sigma}\left(f_{G,local}^{\natural}\right)\ \left(\mathsf{P}_{\mathcal{C}_{G}}^{\omega}g\right)\omega\right| \lesssim \mathfrak{T}\left\|f\right\|_{L^{2}(\sigma)}\left\|g\right\|_{L^{2}(\omega)},$$
$$\left|\sum_{G\in\mathcal{G}}\int H_{\sigma}\left(f_{G,stopping}^{\natural}\right)\ \left(\mathsf{P}_{\mathcal{C}_{G}}^{\omega}g\right)\omega\right| \lesssim \sqrt{\mathcal{A}_{2}}\left\|f\right\|_{L^{2}(\sigma)}\left\|g\right\|_{L^{2}(\omega)},$$
$$\left|\sum_{G\in\mathcal{G}}\int H_{\sigma}\left(f_{G,corona}^{\natural}\right)\ \left(\mathsf{P}_{\mathcal{C}_{G}}^{\omega}g\right)\omega\right| \lesssim \mathfrak{F}\left\|f\right\|_{L^{2}(\sigma)}\left\|g\right\|_{L^{2}(\omega)},$$

now all follow as in the previous proof using properties (1), (2), (3) and (4) above.

Indeed, using (1) and (4) as above, the estimate (4.10) becomes

$$\begin{aligned}\left|\sum_{G\in\mathcal{G}}\int H_{\sigma}\left(f_{G,local}^{\natural}\right)\ \left(\mathsf{P}_{\mathcal{C}_{G}}^{\omega}g\right)\omega\right| &\lesssim \mathfrak{T}\left(\sum_{F\in\mathcal{F}}\alpha_{\mathcal{F}}\left(F\right)^{2}\left|F\right|_{\sigma}\right)^{\frac{1}{2}}\left(\sum_{F\in\mathcal{F}}\left\|\mathsf{R}_{F}^{\omega}g\right\|_{L^{2}(\omega)}^{2}\right)^{\frac{1}{2}}\\ &\lesssim \mathfrak{T}\left\|f\right\|_{L^{2}(\sigma)}\left\|g\right\|_{L^{2}(\omega)},\end{aligned}$$

where in the last line we have used property (3) above, together with the orthogonality of the projections $\mathsf{R}_{F}^{\omega}=\sum_{G\in\mathcal{G}:\ F_{1}(G)=F}\mathsf{P}_{\mathcal{C}_{G}}^{\omega}g$. Then using (1) and (4) again, the estimate (4.12) becomes

$$\left|\sum_{G\in\mathcal{G}}\int H_{\sigma}\left(f_{G,stopping}^{\natural}\right)\ \left(\mathsf{P}_{\mathcal{C}_{G}}^{\omega}g\right)\omega\right|\lesssim\sum_{F\in\mathcal{F}}\alpha_{\mathcal{F}}\left(F\right)\sum_{J\subset\theta(F)}\left|\left\langle H_{\sigma}\left(\mathbf{1}_{F}\right),\triangle_{J}^{\omega}g\right\rangle_{\omega}\right|,$$

which is dominated by $\sqrt{\mathcal{A}_{2}}\left\|f\right\|_{L^{2}(\sigma)}\left\|g\right\|_{L^{2}(\omega)}$ upon application of Lemma 3.16 and property (2) above. Finally, using (1) and (4) yet again, the estimate (4.13) becomes

$$\left|\beta_{F_{n+1},G}\left(x\right)\right|\leq\alpha_{\mathcal{F}}\left(F_{n+1}\right)\mathbf{1}_{F_{n+1}\setminus F_{n}}\left(x\right),$$

and this transforms the estimate (4.14) into

$$\begin{aligned}&\left|\sum_{J\in\mathcal{D}^{\omega}}\int H_{\sigma}\left(f_{G(J),corona}^{\natural}\right)\ \left(\triangle_{J}^{\omega}g\right)\omega\right|\\ &\lesssim \sum_{K\in\mathcal{F}}\sum_{J^{*}\in\mathcal{J}^{*}(K)}\left\langle x,\sum_{\substack{J\in\mathcal{D}^{\omega}:\ J\subset J^{*}\\ \pi_{\mathcal{F}}G(J)=K}}\triangle_{J}^{\omega}\widetilde{g}\right\rangle_{\omega}\frac{1}{\left|J^{*}\right|}\mathsf{P}\left(J^{*},\sum_{F\in\mathcal{F}:\ K\subsetneqq F}\alpha_{\mathcal{F}}\left(\pi_{\mathcal{F}}F\right)\mathbf{1}_{\pi_{\mathcal{F}}F\setminus F}\sigma\right)\\ &\leq \sum_{K\in\mathcal{F}}\sum_{J^{*}\in\mathcal{J}^{*}(K)}\left\langle\frac{x}{\left|J^{*}\right|},\mathbf{1}_{J^{*}}g_{K}\right\rangle_{\omega}\mathsf{P}\left(J^{*},\mathcal{M}_{\alpha}f\right),\end{aligned}$$

where

$$\mathcal{M}_{\alpha}f\equiv\sum_{F\in\mathcal{F}}\alpha_{\mathcal{F}}\left(F\right)\mathbf{1}_{F}$$

substitutes for the maximal function $\mathcal{M}_{\sigma}f$ used earlier. Now we use property (3), together with the fact that the collection of functions $\left\{g_{K}\right\}_{K\in\mathcal{F}}$ is $\mathcal{F}$-adapted as in Definition 2.4, to obtain the bound

$$\sum_{K\in\mathcal{F}}\sum_{J^{*}\in\mathcal{J}^{*}(K)}\left\langle\frac{x}{\left|J^{*}\right|},\mathbf{1}_{J^{*}}g_{K}\right\rangle_{\omega}\mathsf{P}\left(J^{*},\mathcal{M}_{\alpha}f\right)\lesssim\mathfrak{F}\left\|f\right\|_{L^{2}(\sigma)}\left\|g\right\|_{L^{2}(\omega)}.$$

This establishes the bound for the first term on the left side of (4.3) in Proposition 4.4, but with the constant $\mathfrak{NTV}+\mathfrak{F}+\mathfrak{F}^{*}$ on the right side. Proposition 4.22 now applies to complete the proof of Proposition 4.4. The proof for the second term $\mathsf{B}_{mix}(f,g)$ on the left side of (4.3) is also similar to the corresponding proof in Proposition 4.2, using only modifications of the type already described above. Proposition 4.22 now applies to complete the proof of Proposition 4.4.

4.3.1. *Proof of the Iterated Corona Proposition.* We can now prove the Iterated Corona Proposition 1.11. For this we return to the parallel corona splitting (1.7), but with general stopping data for each of f and g. We then decompose the far form $\mathsf{H}_{far}(f,g)$ into lower and upper forms in analogy with $\mathcal{H}_{lower}$ and $\mathcal{H}_{upper}$ in (1.5):

$$\begin{aligned}\mathsf{H}_{far}(f,g) &= \left\{\sum_{\substack{(F,G)\in\mathsf{Far}(\mathcal{F}\times\mathcal{G})\\ G\subset F}}+\sum_{\substack{(F,G)\in\mathsf{Far}(\mathcal{F}\times\mathcal{G})\\ F\subset G}}\right\}\left\langle H_{\sigma}\mathsf{P}^{\sigma}_{\mathcal{C}_F}f,\mathsf{P}^{\omega}_{\mathcal{C}_G}g\right\rangle_{\omega}\\ &\equiv \mathsf{H}_{far\ lower}(f,g)+\mathsf{H}_{far\ upper}(f,g)\,.\end{aligned}$$

Let N_{near}, $\mathsf{N}_{disjoint}$, $\mathsf{N}_{far\ lower}$ and $\mathsf{N}_{far\ upper}$ be the bounds for the nonlinear corona forms $\mathsf{H}_{near}(f,g)$, $\mathsf{H}_{disjoint}(f,g)$, $\mathsf{H}_{far\ lower}(f,g)$ and $\mathsf{H}_{far\ upper}(f,g)$. As mentioned earlier, Lemma 3.1 gives

$$\mathsf{N}_{disjoint}\lesssim\mathfrak{NTV}.$$

Our goal here is to show the inequality

$$\mathsf{N}_{far\ lower}+\mathsf{N}_{far\ upper}\lesssim\mathfrak{NTV}. \tag{4.39}$$

By symmetry, it suffices to consider $\mathsf{N}_{far\ lower}$, and since the form $\mathsf{H}_{far\ lower}(f,g)$ is controlled by the Intertwining Proposition 4.2, we have

$$\mathsf{N}_{far\ lower}+\mathsf{N}_{far\ upper}\lesssim\mathfrak{NTV},$$

which completes the proof of the Iterated Corona Proposition 1.11.

5. Decomposing the functions

We apply three different corona decompositions in succession to the function $f\in L^2(\sigma)$, gaining structure with each application; first to bounded fluctuation for f, then to minimal bounded fluctuation for f, and finally to regularizing the weight σ. The same is done for $g\in L^2(\omega)$. Finally, we combine these decompositions for f and g into a *triple* parallel corona decomposition to which the Iterated Corona Proposition 1.11 and the Intertwining Proposition 4.4 apply.

5.1. **Bounded fluctuation.** The connection between bounded fluctuation and the corona projections $\mathsf{P}^{\sigma}_{\mathcal{C}^{\sigma}_{F}}f$ in the CZ decomposition of f is given in Lemma 5.2 below. We need the following definition. Given $\gamma>1$, an interval $K\in\mathcal{D}^{\sigma}$ and a function f supported on K, we say that f is a γ-*simple* function of *bounded fluctuation* on K, written $f \in \mathcal{SBF}_{\sigma}^{(\gamma)}(K)$, if there is a pairwise disjoint collection $\mathcal{K}$ of $\mathcal{D}^{\sigma}$-subintervals of K such that

$$\begin{aligned} f &= \sum_{K' \in \mathcal{K}} a_{K'} \mathbf{1}_{K'}, \\ a_{K'} &> \gamma, \qquad K' \in \mathcal{K}, \\ \frac{1}{|I|_{\sigma}} \int_I f\sigma &\leq 1, \qquad I \in \widehat{\mathcal{K}}, \end{aligned}$$

where

$$\widehat{\mathcal{K}} = \{I \in \mathcal{D}^{\sigma} : I \subset K \text{ and } I \supsetneqq K' \text{ for some } K' \in \mathcal{K}\}.$$

Using the facts that $\frac{1}{|I|_{\sigma}} \int_I |f|\,\sigma \leq 1$ for $I \in \widehat{\mathcal{K}}$ and $\frac{1}{|I|_{\sigma}} \int_I |f|\,\sigma > \gamma$ for $I \in \mathcal{K}$, it is easy to see that the collection $\mathcal{K}$ is uniquely determined by the simple function f of bounded fluctuation, so we will typically write $\mathcal{K}_f$ for this collection when $f \in \mathcal{SBF}_{\sigma}^{(\gamma)}(K)$. Note that functions in $\mathcal{SBF}_{\sigma}^{(\gamma)}(K)$, unlike those in $\mathcal{BF}_{\sigma}^{(\gamma)}(K)$, do *not* have vanishing mean.

Remark 5.1. *There is a more general notion of simple function of bounded fluctuation on K, that permits f to take on both positive and negative values, namely we say that $f \in \mathcal{GBF}_{\sigma}^{(\gamma)}(K)$ if we only require $|a_{K'}| > \gamma$ for $K' \in \mathcal{K}$ and $\frac{1}{|I|_{\sigma}} \int_I |f|\,\sigma \leq 1$ for $I \in \widehat{\mathcal{K}}$ along with the other restrictions. However, every function in $\mathcal{GBF}_{\sigma}^{(\gamma)}(K)$ can be written as the difference of two functions in $\mathcal{SBF}_{\sigma}^{(\gamma)}(K)$, and it will be a key point in the proof of Proposition 5.8 that all the values of such f can take a single sign.*

Lemma 5.2. *Suppose that $\mathcal{F}$ is a stopping collection for $f \in L^2(\sigma)$ with Calderón-Zygmund stopping constant $C_0 \geq 4$. Given $\gamma > 1$, there is for each $F \in \mathcal{F}$ a decomposition,*

$$(5.3) \qquad \begin{aligned} \mathsf{P}_{\mathcal{C}_F}^{\sigma} f &= \left(\mathsf{P}_{\mathcal{C}_F}^{\sigma} f\right)_1 + \left(\mathsf{P}_{\mathcal{C}_F}^{\sigma} f\right)_2; \\ \left| \frac{1}{(C_0\gamma + \gamma + 1)\,\mathbb{E}_F^{\sigma}|f|} \left(\mathsf{P}_{\mathcal{C}_F}^{\sigma} f\right)_1 \right| &\leq \mathbf{1}_F, \\ \frac{1}{(C_0 + 1)\,\mathbb{E}_F^{\sigma}|f|} \left(\mathsf{P}_{\mathcal{C}_F}^{\sigma} f\right)_2 &\in \mathcal{GBF}_{\sigma}^{(\gamma)}(F). \end{aligned}$$

In particular, we have

$$\begin{aligned} \mathsf{P}_{\mathcal{C}_F}^{\sigma} f &= (C_0\gamma + \gamma + 1)\left(\mathbb{E}_F^{\sigma}|f|\right)\ h_0 + (C_0 + 1)\left(\mathbb{E}_F^{\sigma}|f|\right)\ h_1; \\ h_i &\in \mathcal{BF}_{\sigma}^{(\gamma)}(F), \qquad i = 1, 2. \end{aligned}$$

Proof. To obtain (5.3), fix $F \in \mathcal{F}$ for the moment, and write

$$\mathsf{P}_{\mathcal{C}_F}^{\sigma} f = \mathbf{1}_{\widehat{F}} \mathsf{P}_{\mathcal{C}_F}^{\sigma} f + \mathbf{1}_{\widetilde{F}} \mathsf{P}_{\mathcal{C}_F}^{\sigma} f,$$

where

$$\widehat{F} \equiv F \setminus \widetilde{F} \text{ and } \widetilde{F} \equiv \bigcup_{F' \in \mathfrak{C}(F)} F'.$$

Then if $x \in \widehat{F}$ we have

$$\mathsf{P}_{\mathcal{C}(F)}^{\sigma} f(x) = \sum_{I \in \mathcal{C}_F} \triangle_I^{\sigma} f(x) = \mathbb{E}_{K(x)}^{\sigma} f - \mathbb{E}_F^{\sigma} f,$$

where $K(x)$ is the smallest child of any interval in the corona $\mathcal{C}_F$ that contains x. Thus

$$\left|\mathsf{P}^{\sigma}_{\mathcal{C}_F} f(x)\ \mathbf{1}_{\widehat{F}}(x)\right| = \left|\mathbb{E}^{\sigma}_{K(x)} f - \mathbb{E}^{\sigma}_{F} f\right|\ \mathbf{1}_{\widehat{F}}(x) \leq (C_0+1)\ \mathbb{E}^{\sigma}_{F}|f|\ \mathbf{1}_{\widehat{F}}(x), \tag{5.4}$$

where C_0 is the Calderón-Zygmund stopping constant, and

$$\mathbf{1}_{\widehat{F}} \mathsf{P}^{\sigma}_{\mathcal{C}_F} f = \sum_{F' \in \mathfrak{C}(F)} \left(\mathbb{E}^{\sigma}_{F'} f - \mathbb{E}^{\sigma}_{F} f\right)\ \mathbf{1}_{F'}.$$

Now let

$$\mathfrak{C}_{big}(F) = \left\{F' \in \mathfrak{C}(F) : \left|\mathbb{E}^{\sigma}_{F'} f - \mathbb{E}^{\sigma}_{F} f\right| > (C_0\gamma + \gamma + 1)\, \mathbb{E}^{\sigma}_{F}|f|\right\},$$

set $\mathfrak{C}_{small}(F) = \mathfrak{C}(F) \setminus \mathfrak{C}_{big}(F)$, and then define

$$\begin{aligned}
\left(\mathsf{P}^{\sigma}_{\mathcal{C}_F} f\right)_1 &= \mathbf{1}_{\widehat{F}} \mathsf{P}^{\sigma}_{\mathcal{C}_F} f(x) + \sum_{F' \in \mathfrak{C}_{small}(F)} \left(\mathbb{E}^{\sigma}_{F'} f - \mathbb{E}^{\sigma}_{F} f\right)\ \mathbf{1}_{F'}, \\
\left(\mathsf{P}^{\sigma}_{\mathcal{C}_F} f\right)_2 &= \sum_{F' \in \mathfrak{C}_{big}(F)} \left(\mathbb{E}^{\sigma}_{F'} f - \mathbb{E}^{\sigma}_{F} f\right)\ \mathbf{1}_{F'},
\end{aligned}$$

to obtain the decomposition (5.3).

Indeed, from (5.4) and the definition of $\mathfrak{C}_{small}(F)$ we have

$$\begin{aligned}
\left|\left(\mathsf{P}^{\sigma}_{\mathcal{C}_F} f\right)_1\right| &\leq \max\left\{\left\|\mathsf{P}^{\sigma}_{\mathcal{C}_F} f(x)\ \mathbf{1}_{\widehat{F}}(x)\right\|_{L^\infty(\sigma)}, \sup_{F' \in \mathfrak{C}_{small}(F)} \left|\mathbb{E}^{\sigma}_{F'} f - \mathbb{E}^{\sigma}_{F} f\right|\right\} \\
&\leq (C_0\gamma + \gamma + 1)\, \mathbb{E}^{\sigma}_{F}|f|.
\end{aligned}$$

To see that $\frac{1}{(C_0+1)\mathbb{E}^{\sigma}_{F}|f|}\left(\mathsf{P}^{\sigma}_{\mathcal{C}_F} f\right)_2 \in \mathcal{GBF}_{\sigma}(F)$, take $I \subset F$ such that $I \supsetneqq F'$ for some $F' \in \mathfrak{C}_{big}(F)$. Then we have

$$\begin{aligned}
\frac{1}{|I|_\sigma}\int_I \left|\left(\mathsf{P}^{\sigma}_{\mathcal{C}_F} f\right)_2\right| \sigma &= \frac{1}{|I|_\sigma}\int_I \left|\sum_{F' \in \mathfrak{C}_{big}(F):\ F' \subset I} \left(\mathbb{E}^{\sigma}_{F'} f - \mathbb{E}^{\sigma}_{F} f\right)\ \mathbf{1}_{F'}\right| \sigma \\
&\leq \mathbb{E}^{\sigma}_{F}|f| + \frac{1}{|I|_\sigma}\int_I \left(\sum_{F' \in \mathfrak{C}_{big}(F):\ F' \subset I} \left(\mathbb{E}^{\sigma}_{F'}|f|\right)\ \mathbf{1}_{F'}\right)\sigma \\
&\leq \mathbb{E}^{\sigma}_{F}|f| + \frac{1}{|I|_\sigma}\int_I |f|\,\sigma \\
&\leq \mathbb{E}^{\sigma}_{F}|f| + \frac{1}{|I|_\sigma} C_0 \mathbb{E}^{\sigma}_{F}|f|\, |I|_\sigma \leq (C_0+1)\, \mathbb{E}^{\sigma}_{F}|f|,
\end{aligned}$$

where C_0 is the Calderón-Zygmund stopping constant. On the other hand, for $F' \in \mathfrak{C}_{big}(F)$, we have

$$\begin{aligned}
\frac{1}{|F'|_\sigma}\int_{F'} \left|\left(\mathsf{P}^{\sigma}_{\mathcal{C}_F} f\right)_2\right| \sigma &= \left|\mathbb{E}^{\sigma}_{F'} f - \mathbb{E}^{\sigma}_{F} f\right| > (C_0\gamma + \gamma + 1)\, \mathbb{E}^{\sigma}_{F}|f| - \mathbb{E}^{\sigma}_{F}|f| \\
&= (C_0+1)\left(\mathbb{E}^{\sigma}_{F}|f|\right)\gamma.
\end{aligned}$$

□

5.2. **Minimal bounded fluctuation.** In order to continue the proof of Theorem 1, we must make a crucial decomposition of functions $f \in \mathcal{BF}_{\sigma}(K)$ into bounded functions and functions of *minimal* bounded fluctuation, the latter functions having a great deal of additional structure owing to their minimal Haar support. We will present the decomposition in three stages, first to bounded and *simple* functions of bounded fluctuation, then to bounded and *prebounded* and *prefluctuation* functions, and finally to bounded and functions of *minimal* bounded fluctuation.

We begin by recalling from Definition 2.1 that $f \in \mathcal{BF}_{\sigma}(K)$ if it is supported in K with mean zero, and equals a constant $a_{K'}$ of modulus greater than γ on any subinterval K' where $\mathbb{E}^{\sigma}_{K'}|f| > 1$. If we require in addition that

$$a_{K'} > \gamma, \qquad K' \in \mathcal{K}_f,$$

then we denote the resulting collection of functions by $\mathcal{PBF}^{(\gamma)}_{\sigma}(K)$. Recall also that $\mathcal{SBF}^{(\gamma)}_{\sigma}(K)$ consists of those functions $f \in \mathcal{PBF}_{\sigma}(K)$ for which $f = \sum_{K' \in \mathcal{K}_f} a_{K'} \mathbf{1}_{K'}$. We have the following simple decomposition.

Lemma 5.5. *Suppose that $f \in \mathcal{BF}^{(\gamma)}_{\sigma}(K)$. Then we can write*

$$f = h_{bdd} + h^{+}_{fluc} - h^{-}_{fluc},$$

where $h_{bdd} \in (L^{\infty}_{K})_1(\sigma)$ and $h^{\pm}_{fluc} \in \mathcal{SBF}^{(\gamma)}_{\sigma}(K)$.

Proof. We simply define

$$\begin{aligned} h^{+}_{fluc} &\equiv \sum_{K' \in \mathcal{K}_f:\ a_{K'} > \gamma} a_{K'} \mathbf{1}_{K'}, \\ h^{-}_{fluc} &\equiv \sum_{K' \in \mathcal{K}_f:\ a_{K'} < \gamma} a_{K'} \mathbf{1}_{K'}, \\ h_{bdd} &\equiv f \mathbf{1}_{K \setminus \bigcup_{K' \in \mathcal{K}_f} K'}. \end{aligned}$$

□

We now prepare to give our main proposition on decomposing a function of bounded fluctuation into a sum of bounded and *minimal* bounded fluctuation functions, which we refer to as restricted bounded fluctuation functions. We set

$$\begin{aligned} L^{\infty}_{F}(\sigma) &\equiv \left\{ f \in L^{\infty}(\sigma) : supp\ f \subset F \text{ and } \int_K f d\sigma = 0 \right\}, \\ L^{\infty}_{F}(\sigma)_1 &\equiv \left\{ f \in L^{\infty}_{F}(\sigma) : \|f\|_{L^{\infty}(\sigma)} \leq 1 \right\}. \end{aligned}$$

Definition 5.6. *Define the set of functions $\mathcal{RBF}^{(\gamma)}_{\sigma}(F)$ of* restricted *bounded fluctuation on F by*

$$\mathcal{RBF}^{(\gamma)}_{\sigma}(F) \equiv \mathcal{MBF}^{(\gamma)}_{\sigma}(F) + L^{\infty}_{F}(\sigma)_1 \ .$$

Next, we record a decomposition of f into prebounded and prefluctuation functions in part (1) of the proposition.

Definition 5.7. *Let $\gamma \geq 4$. A function f supported on an interval $K \in \mathcal{D}$ is a* prebounded *function on K if*

$$\|\triangle^{\sigma}_{I} f\|_{\infty} \leq 4, \qquad \text{for all } I \in \mathcal{D}.$$

A function f supported on an interval $K \in \mathcal{D}$ is a prefluctuation^{+} *function on K, respectively a* prefluctuation^{-} *function on K, (relative to γ) if*

$$\sup_I \triangle_I^\sigma f > \gamma \text{ and } \mathbb{E}_I^\sigma \left|\triangle_I^\sigma f\right| \leq 2, \qquad \text{for all } I \text{ such that } \widehat{f}(I) \neq 0,$$

respectively

$$\inf_I \triangle_I^\sigma f_{fluc}^{+} < -\gamma \text{ and } \mathbb{E}_I^\sigma \left|\triangle_I^\sigma f\right| \leq 2, \qquad \text{for all } I \text{ such that } \widehat{f}(I) \neq 0.$$

The point of these definitions is that the following properties hold: the Calderón-Zygmund decomposition of a prebounded function has corona projections that are *bounded*, and the Calderón-Zygmund decomposition of a prefluctuation function has corona projections that are of *restricted bounded fluctuation*. In general, neither of these properties hold for Calderón-Zygmund corona projections of general functions when the measure σ is nondoubling.

Finally, we note that our decomposition below is *infinite*, and necessarily so by the example in the appendix.

Proposition 5.8. *Suppose that $f \in \mathcal{BF}_\sigma^{(\gamma)}(K)$ as in Definition 2.1 with $\gamma \geq 16$.*

(1) *There is a decomposition*

$$f = f_{bdd} + f_{fluc}^{+} - f_{fluc}^{-}, \tag{5.9}$$

of f into prebounded and prefluctuation$^{\pm}$ functions $f_{bdd}, f_{fluc}^{+}, f_{fluc}^{-}$ on K, i.e.

$$\begin{aligned} \left\|\triangle_I^\sigma f_{bdd}\right\|_\infty &\leq 4, \qquad I \in \mathcal{D}, \\ \sup_I \triangle_I^\sigma f_{fluc}^{+} &> \gamma \text{ and } \mathbb{E}_I^\sigma \left|\triangle_I^\sigma f_{fluc}^{+}\right| \leq 2, \qquad \widehat{f_{fluc}^{+}}(I) \neq 0, \\ \inf_I \triangle_I^\sigma f_{fluc}^{-} &< -\gamma \text{ and } \mathbb{E}_I^\sigma \left|\triangle_I^\sigma f_{fluc}^{-}\right| \leq 2, \qquad \widehat{f_{fluc}^{-}}(I) \neq 0. \end{aligned}$$

(2) *There are collections of stopping data for f_{bdd} and $f_{fluc}^{\pm}$, with stopping times $\mathcal{S}$ and $\mathcal{T}^{\pm}$, and corona projections $f_{bdd,S} \equiv \mathsf{P}_{\mathcal{C}_S}^\sigma f_{bdd}$ and $f_{fluc,T}^{\pm} \equiv \mathsf{P}_{\mathcal{C}_{T^{\pm}}}^\sigma f_{fluc}^{\pm}$, that satisfy Carleson conditions*

$$\begin{aligned} \sum_{S' \subset S} \left|S'\right|_\sigma &\leq 4\left|S\right|_\sigma, \qquad S \in \mathcal{S}, \\ \sum_{T' \subset T} \left|T'\right|_\sigma &\leq 4\left|T\right|_\sigma, \qquad T \in \mathcal{T}^{\pm}, \end{aligned}$$

and a quasi-orthogonal decomposition

$$\begin{aligned} f &= f_{bdd} + f_{fluc}^{+} - f_{fluc}^{-}, \\ &= \sum_{S \in \mathcal{S}} f_{bdd,S} + \sum_{T \in \mathcal{T}^{+}} f_{fluc,T}^{+} - \sum_{T \in \mathcal{T}^{-}} f_{fluc,T}^{-}, \\ \left\|f\right\|_{L^2(\sigma)}^2 &\approx \sum_{S \in \mathcal{S}} \left\|f_{bdd,S}\right\|_{L^2(\sigma)}^2 + \sum_{T \in \mathcal{T}^{+}} \left\|f_{fluc,T}^{+}\right\|_{L^2(\sigma)}^2 + \sum_{T \in \mathcal{T}^{-}} \left\|f_{fluc,T}^{-}\right\|_{L^2(\sigma)}^2, \end{aligned} \tag{5.10}$$

such that for all $S\in\mathcal{S}$ and $T\in\mathcal{T}^{\pm}$,

$$\left\Vert \frac{1}{5\mathbb{E}_{S}^{\sigma}\left\vert f_{bdd}\right\vert }f_{bdd,S}\right\Vert _{L^{\infty}(S)} \leq 1, \tag{5.11}$$
$$\frac{1}{30\mathbb{E}_{T}^{\sigma}\left\vert f_{fluc}^{\pm}\right\vert }f_{fluc,T}^{\pm} \in \mathcal{RBF}_{\sigma}^{\left(\frac{1}{2}\right)}\left(T\right).$$

Note that we do not assert any control on the averages $\mathbb{E}_{K}^{\sigma}\left\vert f_{bdd}\right\vert$ and $\mathbb{E}_{K}^{\sigma}\left\vert f_{fluc}^{\pm}\right\vert$ in Proposition 5.8. The quasi-orthogonality in (5.10) is, together with (5.11) and the Carleson conditions, sufficient to adequately control the $L^{2}\left(\sigma\right)$ norm of f.

Proof. We begin by applying Lemma 5.5 to obtain a splitting

$$f=h_{bdd}+h_{fluc}^{+}-h_{fluc}^{-},$$

where $\left\Vert h_{bdd}\right\Vert_{\infty}\leq 1$ and $h_{fluc}^{\pm}\in\mathcal{SBF}_{\sigma}^{(\gamma)}\left(K\right)$. If we write

$$h_{fluc}^{\pm}=\left(\mathbb{E}_{K}^{\sigma}h_{fluc}^{\pm}\right)\mathbf{1}_{K}+g_{fluc}^{\pm},$$

then $g_{fluc}^{\pm}\in\mathcal{PBF}_{\sigma}^{\left(\frac{\gamma-1}{2}\right)}\left(K\right)$ provided $\gamma>1$ is chosen large enough. Fix a sign $\pm$ and for convenience write $g=g_{fluc}^{\pm}$ momentarily. Since $\int_{K}g\sigma=0$ and g is constant on each $K'\in\mathcal{K}_{g}$, we have

$$supp\ \widehat{g}\subset\widehat{\mathcal{K}_{g}}\equiv\left\{I\subset K:I\not\subset K'\text{ for any }K'\in\mathcal{K}_{g}\right\}.$$

Now we split $\widehat{\mathcal{K}_{g}}$ into two pairwise disjoint subsets:

$$\begin{aligned}\widehat{\mathcal{K}_{g}} &= \left\{I\in\widehat{\mathcal{K}_{g}}:\left\Vert\triangle_{I}^{\sigma}g\right\Vert_{\infty}\leq\frac{\gamma-1}{2}\right\}\dot{\bigcup}\left\{I\in\widehat{\mathcal{K}_{g}}:\sup_{I}\triangle_{I}^{\sigma}g>\frac{\gamma-1}{2}\right\}\\ &\equiv \widehat{\mathcal{K}_{g}}\left(bounded\right)\dot{\bigcup}\widehat{\mathcal{K}_{g}}\left(\text{positive}\right).\end{aligned}$$

If we write I_{small} and I_{large} for the two children of I where $\left\vert I_{small}\right\vert_{\sigma}\leq\left\vert I_{large}\right\vert_{\sigma}$, then we have

$$\triangle_{I}^{\sigma}\ g=\left(\mathbb{E}_{I_{small}}^{\sigma}g-\mathbb{E}_{I}^{\sigma}g\right)\mathbf{1}_{I_{small}}+\left(\mathbb{E}_{I_{large}}^{\sigma}g-\mathbb{E}_{I}^{\sigma}g\right)\mathbf{1}_{I_{large}}, \tag{5.12}$$

where

$$\mathbb{E}_{I}^{\sigma}\left\vert g\right\vert\leq 1\text{ and }\mathbb{E}_{I_{large}}^{\sigma}\left\vert g\right\vert\leq 1\text{ for }I\in\widehat{\mathcal{K}_{g}},$$

since $g\in\mathcal{BF}_{\sigma}^{\left(\frac{\gamma-1}{2}\right)}\left(K\right)$. It follows that if $\left\Vert\triangle_{I}^{\sigma}g\right\Vert_{\infty}\leq\frac{\gamma-1}{2}$, then in fact we have the better bound $\left\Vert\triangle_{I}^{\sigma}g\right\Vert_{\infty}\leq 2$, so that

$$\left\Vert\triangle_{I}^{\sigma}g\right\Vert_{\infty}\leq 2,\qquad I\in\widehat{\mathcal{K}_{g}}\left(bounded\right). \tag{5.13}$$

It also follows that if I belongs to $\widehat{\mathcal{K}_{g}}$ (positive), then

$$\frac{\gamma-3}{2}<\left\vert\mathbb{E}_{I_{small}}^{\sigma}g\right\vert\leq\mathbb{E}_{I_{small}}^{\sigma}\left\vert g\right\vert\leq\frac{\left\vert I\right\vert_{\sigma}}{\left\vert I_{small}\right\vert_{\sigma}}\mathbb{E}_{I}^{\sigma}\left\vert g\right\vert\leq\frac{\left\vert I\right\vert_{\sigma}}{\left\vert I_{small}\right\vert_{\sigma}},$$

which shows that $I_{small}\in\mathcal{K}_{g}$, i.e. $I\in\pi\mathcal{K}_{g}$, and in addition that

$$\left\vert I_{small}\right\vert_{\sigma}<\frac{2}{\gamma-3}\left\vert I\right\vert_{\sigma}.$$

Now recalling that $g = g^{\pm}_{fluc}$, we define

$$\begin{aligned} f^{\pm}_{bdd} &= \sum_{I \in \widehat{\mathcal{K}_{g^{\pm}_{fluc}}}(bounded)} \triangle^{\sigma}_I g^{\pm}_{fluc}, \\ f^{\pm}_{fluc} &= \sum_{I \in \widehat{\mathcal{K}_{g^{\pm}_{fluc}}}(\text{positive})} \triangle^{\sigma}_I g^{\pm}_{fluc}, \\ f_{bdd} &= h_{bdd} + \left(\mathbb{E}^{\sigma}_K h^{+}_{fluc}\right) \mathbf{1}_K - \left(\mathbb{E}^{\sigma}_K h^{-}_{fluc}\right) \mathbf{1}_K + f^{+}_{bdd} - f^{-}_{bdd}, \end{aligned}$$

and note that so far we have shown

$$\begin{aligned} \|\triangle^{\sigma}_I f_{bdd}\|_{\infty} &\leq \left\|\triangle^{\sigma}_I \left[h_{bdd} + \left(\mathbb{E}^{\sigma}_K h^{+}_{fluc}\right) \mathbf{1}_K - \left(\mathbb{E}^{\sigma}_K h^{-}_{fluc}\right) \mathbf{1}_K\right]\right\|_{\infty} \\ &\quad + \left\|\triangle^{\sigma}_I \left(f^{+}_{bdd} - f^{-}_{bdd}\right)\right\|_{\infty} \\ &\leq 6 + 4 = 10, \qquad I \in \mathcal{D}, \\ supp\ \widehat{f^{\pm}_{fluc}} &= \widehat{\mathcal{K}_{g^{\pm}_{fluc}}}(\text{positive}) = \pi \mathcal{K}_{g^{\pm}_{fluc}}, \end{aligned} \tag{5.14}$$

for $\gamma > 1$ large enough. This establishes (5.9).

Now we apply a standard Calderón-Zygmund decomposition to f_{bdd} to obtain stopping times $\mathcal{S}$ with top interval $S_0 = K$ and

$$\begin{aligned} Child\,(S_0) &\equiv \left\{S \in \mathcal{D} : S \subset S_0 = K \text{ is maximal w.r.t. } \mathbb{E}^{\sigma}_S |f_{bdd}| > 4\mathbb{E}^{\sigma}_{S_0} |f_{bdd}|\right\}, \\ Child\,(S) &\equiv \left\{S' \in \mathcal{D} : S' \subset S \text{ is maximal w.r.t. } \mathbb{E}^{\sigma}_{S'} |f_{bdd}| > 4\mathbb{E}^{\sigma}_S |f_{bdd}|\right\}. \end{aligned}$$

We then have

$$f_{bdd} = \sum_{S \in \mathcal{S}} \mathsf{P}^{\sigma}_{\mathcal{C}_S} f_{bdd}.$$

Now comes the first crucial point. The functions $\mathsf{P}^{\sigma}_{\mathcal{C}_S} f_{bdd}$ are *bounded* by $5\mathbb{E}^{\sigma}_S |f_{bdd}|$ if $S \in \mathcal{S}$. Indeed, with the notation $\widetilde{S} \equiv \bigcup_{S' \in Child(S)} S'$ and $\widehat{S} \equiv S \setminus \widetilde{S}$, we have for $S \neq S_0$,

$$\begin{aligned} \left\|\mathsf{P}^{\sigma}_{\mathcal{C}_S} f_{bdd}\right\|_{\infty} &\leq \max\left\{\left\|\mathbf{1}_{\widehat{S}} \mathsf{P}^{\sigma}_{\mathcal{C}_S} f_{bdd}\right\|_{\infty}, \left\|\mathbf{1}_{\widetilde{S}} \mathsf{P}^{\sigma}_{\mathcal{C}_S} f_{bdd}\right\|_{\infty}\right\} \\ &\leq \max\left\{4\mathbb{E}^{\sigma}_S |f_{bdd}|, \sup_{S' \in Child(S)} \left|\mathbb{E}^{\sigma}_{S'} f_{bdd} - \mathbb{E}^{\sigma}_S f_{bdd}\right|\right\} \leq 5\mathbb{E}^{\sigma}_S |f_{bdd}|, \end{aligned}$$

since (5.12) and (5.13) give

$$\left|\mathbb{E}^{\sigma}_{S'} f_{bdd}\right| \leq \left\|\triangle^{\sigma}_{\pi S'} f_{bdd}\right\|_{\infty} + \left|\mathbb{E}^{\sigma}_{\pi S'} f_{bdd}\right| \leq 1 + 4\mathbb{E}^{\sigma}_S |f_{bdd}|.$$

This completes the proof of the first half of (5.11).

Now we turn to the function f^{+}_{fluc} and apply a standard Calderón-Zygmund decomposition to f^{+}_{fluc} and obtain stopping times $\mathcal{T}^{+}$ and coronas $\{\mathcal{C}_T\}_{T \in \mathcal{T}^{+}}$ with top stopping interval $T_0 = K$ such that

$$f^{+}_{fluc} = \sum_{T \in \mathcal{T}^{+}} \mathsf{P}^{\sigma}_{\mathcal{C}_T} f^{+}_{fluc}.$$

Let $T \in \mathcal{T}^{+}$ and let $Child_{\mathcal{T}^{+}}(T)$ be the collection of $\mathcal{T}^{+}$-children T' of T. We set

$$\mathcal{C}^{*}_T \equiv \left\{I \in \mathcal{C}_T : \widehat{f^{+}_{fluc}}(I) \neq 0\right\} = \mathcal{C}_T \cap \pi \mathcal{K}_{g^{+}_{fluc}},$$

and denote by

$$\mathfrak{C}_{g^+_{fluc}}(T) \equiv \mathcal{K}_{g^+_{fluc}} \cap Child_{\mathcal{T}^+}(T),$$

those $\mathcal{T}^+$-children $T' \in Child_{\mathcal{T}^+}(T)$ belonging to $\mathcal{K}_{g^+_{fluc}}$. Now set

$$\mathfrak{C}^{(\beta)}_{g^+_{fluc}}(T) \equiv \left\{ T' \in \mathfrak{C}_{g^+_{fluc}}(T) : \mathbb{E}^\sigma_{T'} \triangle^\sigma_{\pi T'} f > (\beta+5)\, \mathbb{E}^\sigma_T \left| f^+_{fluc} \right| \right\}$$

and define

$$\psi^1_T \equiv \sum_{T' \in \mathfrak{C}^{(\beta)}_{g^+_{fluc}}(T)} \triangle^\sigma_{\pi T'} f$$

to be the Haar projection of f onto $\pi \mathfrak{C}^{(\beta)}_{g^+_{fluc}}(T)$. To orient the reader, we point out that there are four pairwise disjoint classes of intervals K' in $\mathcal{K}_{g^+_{fluc}}$ that are subsets of T:

$$\begin{aligned} Class_T(1) &\equiv \left\{ K' \in \mathcal{K}_{g^+_{fluc}} : K' \in \mathcal{C}_T \right\}, \\ Class_T(2) &\equiv \mathfrak{C}_{g^+_{fluc}}(T) \setminus \mathfrak{C}^{(\beta)}_{g^+_{fluc}}(T), \\ Class_T(3) &\equiv \mathfrak{C}^{(\beta)}_{g^+_{fluc}}(T), \\ Class_T(4) &\equiv \left\{ K' \in \mathcal{K}_{g^+_{fluc}} : K' \subsetneqq T' \text{ for some } T' \in Child_{\mathcal{T}^+}(T) \right\}. \end{aligned}$$

For those $K' \in Class_T(1)$, we have

$$\left| \mathbb{E}^\sigma_{K'} \left(\mathsf{P}^\sigma_{\mathcal{C}_T} f^+_{fluc} \right) \right| = \left| \mathbb{E}^\sigma_{K'} f^+_{fluc} - \mathbb{E}^\sigma_T f^+_{fluc} \right| \leq 5 \mathbb{E}^\sigma_T \left| f^+_{fluc} \right|, \tag{5.15}$$

and for those $K' \in Class_T(2)$, we have

$$\begin{aligned} \left| \mathbb{E}^\sigma_{K'} \left(\mathsf{P}^\sigma_{\mathcal{C}_T} f^+_{fluc} \right) \right| &\leq \left| \mathbb{E}^\sigma_{K'} \triangle^\sigma_{\pi K'} f \right| + \left| \mathbb{E}^\sigma_{\pi K'} f^+_{fluc} - \mathbb{E}^\sigma_T f^+_{fluc} \right| \\ &\leq (\beta+5)\, \mathbb{E}^\sigma_T \left| f^+_{fluc} \right| + 5 \mathbb{E}^\sigma_T \left| f^+_{fluc} \right|. \end{aligned} \tag{5.16}$$

When $K' \in Class_T(4)$, the Haar projections $\triangle^\sigma_{\pi T'} f$ are not included in the Haar support of $\mathsf{P}^\sigma_{\mathcal{C}_T} f^+_{fluc}$. Thus it is the $K' \in Class_T(3) = \mathfrak{C}^{(\beta)}_{g^+_{fluc}}(T)$ that can arise as the distinguished intervals for a restricted bounded fluctuation function, and this is what motivates the definition of ψ^1_T above.

Suppose I is an interval in the dyadic grid $\mathcal{D}^\sigma$ that is *not* contained in any K' in $Class_T(1) \cup \mathfrak{C}_{g^+_{fluc}}(T)$. We first note that

$$\mathbb{E}^\sigma_I \left| f^+_{fluc} \right| \leq 4 \mathbb{E}^\sigma_T \left| f^+_{fluc} \right|.$$

Indeed, let L be the smallest interval in the corona $\mathcal{C}_T$ that contains I. From our choice of I, it follows that either $I = L \in \mathcal{C}_T$ and

$$\mathbb{E}^\sigma_I \left| f^+_{fluc} \right| = \mathbb{E}^\sigma_L \left| f^+_{fluc} \right| \leq 4 \mathbb{E}^\sigma_T \left| f^+_{fluc} \right|,$$

or the child L_I of L that contains I is also in the corona $\mathcal{C}_T$ and

$$\mathbb{E}^\sigma_I \left| f^+_{fluc} \right| = \mathbb{E}^\sigma_{L_I} \left| f^+_{fluc} \right| \leq 4 \mathbb{E}^\sigma_T \left| f^+_{fluc} \right|.$$

Set $L^* = L$ or L_I according to whether or not $I \in \mathcal{C}_T$, and note that $\mathbb{E}^\sigma_I \left| \psi^1_T \right| = \mathbb{E}^\sigma_{L^*} \left| \psi^1_T \right|$.

Now comes the second crucial point. The definition of f_{fluc}^{+} implies the inequalities

$$\mathbb{E}_{K'}^{\sigma} \bigtriangleup_{\pi K'}^{\sigma} f > \frac{\gamma - 1}{2} \text{ and } \mathbb{E}_{\theta K'}^{\sigma} \bigtriangleup_{\pi K'}^{\sigma} f < 0,$$

as $\bigtriangleup_{\pi K'}^{\sigma} f$ has mean zero. Thus the expectations $\mathbb{E}_{L^*}^{\sigma} \bigtriangleup_{\pi K'}^{\sigma} f$ all have the same sign when $L^* \subset \theta K'$, and we conclude from our choice of I that

$$\begin{aligned}
\mathbb{E}_I^{\sigma} \left|\psi_T^1\right| &= \mathbb{E}_{L^*}^{\sigma} \left| \sum_{T' \in \mathcal{T}_T^+ :\ L^* \subset \theta T'} \bigtriangleup_{\pi T'}^{\sigma} f \right| = \left| \sum_{T' \in \mathcal{T}_T^+ :\ L^* \subset \theta T'} \mathbb{E}_{L^*}^{\sigma} \bigtriangleup_{\pi T'}^{\sigma} f \right| \\
&\leq \left| \sum_{K' \in Class_T(1) \cup \mathfrak{C}_{g_{fluc}^+}(T) :\ L^* \subset \theta K'} \mathbb{E}_{L^*}^{\sigma} \bigtriangleup_{\pi K'}^{\sigma} f \right| \\
&= \left| \mathbb{E}_{L^*}^{\sigma} \mathsf{P}_{\mathcal{C}_T}^{\sigma} f_{fluc}^{+} \right| = \left| \mathbb{E}_{L^*}^{\sigma} f_{fluc}^{+} - \mathbb{E}_T^{\sigma} f_{fluc}^{+} \right| \leq 5 \mathbb{E}_T^{\sigma} \left| f_{fluc}^{+} \right| .
\end{aligned}$$

Moreover, for $K' \in Class_T(1) \cup Class_T(2)$, we showed in (5.15) and (5.16) above that

$$\left| \mathbb{E}_{K'}^{\sigma} \left(\mathsf{P}_{\mathcal{C}_T}^{\sigma} f_{fluc}^{+} \right) \right| \leq (\beta + 10)\, \mathbb{E}_T^{\sigma} \left| f_{fluc}^{+} \right| .$$

Finally, for $K' \in Class_T(3) \equiv \mathfrak{C}_{g_{fluc}^+}^{(\beta)}(T)$, we have

$$\begin{aligned}
\mathbb{E}_{K'}^{\sigma} \left|\psi_T^1\right| &= \mathbb{E}_{K'}^{\sigma} \left| \sum_{T' \in \mathfrak{C}_{g_{fluc}^+}^{(\beta)}(T)} \bigtriangleup_{\pi T'}^{\sigma} f \right| = \left| -\mathbb{E}_{K'}^{\sigma} \left(\bigtriangleup_{\pi K'}^{\sigma} f \right) + \sum_{T' \in \mathfrak{C}_{g_{fluc}^+}^{(\beta)}(T) :\ K' \subset \theta T'} \bigtriangleup_{\pi T'}^{\sigma} f \right| \\
&\geq \left| \mathbb{E}_{K'}^{\sigma} \left(\bigtriangleup_{\pi K'}^{\sigma} f \right) \right| - \left| \sum_{T' \in \mathfrak{C}_{g_{fluc}^+}^{(\beta)}(T) :\ K' \subset \theta T'} \bigtriangleup_{\pi T'}^{\sigma} f \right| \\
&> (\beta + 5)\, \mathbb{E}_T^{\sigma} \left| f_{fluc}^{+} \right| - 5 \mathbb{E}_T^{\sigma} \left| f_{fluc}^{+} \right| = \beta \mathbb{E}_T^{\sigma} \left| f_{fluc}^{+} \right| .
\end{aligned}$$

It follows that $\frac{1}{(\beta+10)\mathbb{E}_T^{\sigma}\left|f_{fluc}^{+}\right|} \psi_T^1 \in \mathcal{MBF}_{\sigma}^{\left(\frac{\beta}{\beta+10}\right)}(T)$, and the choice $\beta = 20$ gives

$$\frac{1}{30 \mathbb{E}_T^{\sigma} \left| f_{fluc}^{+} \right|} \psi_T^1 \in \mathcal{MBF}_{\sigma}^{\left(\frac{1}{2}\right)}(T) .$$

Now we define

$$\psi_T^0 = \mathsf{P}_{\mathcal{C}_T}^{\sigma} f_{fluc}^{+} - \psi_T^1 = \sum_{K' \in Class_T(1) \cup Class_T(2)} \bigtriangleup_{\pi K'}^{\sigma} f,$$

and note that by the above arguments we have

$$\left\| \psi_T^0 \right\|_{\infty} \leq (\beta + 10)\, \mathbb{E}_T^{\sigma} \left| f_{fluc}^{+} \right| = 30 \mathbb{E}_T^{\sigma} \left| f_{fluc}^{+} \right| ,$$

with the choice $\beta = 20$. Thus we have

$$\begin{aligned}\frac{1}{30\mathbb{E}_T^\sigma \left|f_{fluc}^+\right|} f_{fluc,T}^+ &= \frac{1}{30\mathbb{E}_T^\sigma \left|f_{fluc}^+\right|} \mathsf{P}_{\mathcal{C}_{T^+}}^\sigma f_{fluc}^+ \\ &= \frac{1}{30\mathbb{E}_T^\sigma \left|f_{fluc}^+\right|} \psi_T^0 + \frac{1}{30\mathbb{E}_T^\sigma \left|f_{fluc}^+\right|} \psi_T^1 \\ &\in L_T^\infty (\sigma)_1 + \mathcal{MBF}_\sigma^{\left(\frac{1}{2}\right)} (T) = \mathcal{RBF}_\sigma^{\left(\frac{1}{2}\right)} (T),\end{aligned}$$

which proves the second half of (5.11), and this completes the proof of Proposition 5.8. □

The following lemma is also needed in the sequel.

Lemma 5.17. *Suppose that $\mathcal{C}_T$ is a* connected *grid with top interval $T \subset F$. If f lies in $\mathcal{BF}_\sigma^{(\gamma)}(F)$ (respectively $\mathcal{RBF}_\sigma^{(\gamma)}(F)$) with $\gamma > 0$, then the Haar projection $\frac{1}{2}\mathsf{P}_{\mathcal{C}_T}^\sigma f$ of $\frac{1}{2}f$ lies in $\mathcal{BF}_\sigma^{\left(\frac{\gamma}{2}\right)}(F)$ (respectively $\mathcal{RBF}_\sigma^{\left(\frac{\gamma}{2}\right)}(T)$). If $\mathcal{C}_T$ is an* arbitrary *grid with top interval $T \subset F$, and $f \in \mathcal{MBF}_\sigma^{(\gamma)}(F)$, then $\frac{1}{2}\mathsf{P}_{\mathcal{C}_T}^\sigma f \in \mathcal{MBF}_\sigma^{\left(\frac{\gamma}{2}\right)}(T)$.*

Proof. We prove the assertion for $\mathcal{RBF}_\sigma^{(\gamma)}(F)$, and leave the similar case of $\mathcal{BF}_\sigma^{(\gamma)}(F)$ to the reader. We may assume that either $f \in \mathcal{MBF}_\sigma^{(\gamma)}(F)$ or $f \in (L_F^\infty)_1(\sigma)$. Let $\mathcal{F}$ be the connected hull of the Haar support of f (i.e. the smallest connected grid containing the Haar support of f). In the case that $f \in \mathcal{MBF}_\sigma^{(\gamma)}(F)$, let $\mathcal{K}_f$ be the intervals on which f takes a large constant value, while in the case $f \in (L_F^\infty)_1(\sigma)$, let $\mathcal{K}_f = \emptyset$. The function $\mathsf{P}_{\mathcal{C}_T}^\sigma f$ is supported in T, and will have constant value greater than γ on any interval $F' \in \mathcal{K}_f$ whose parent $\pi F'$ lies in $\mathcal{C}_T$. Otherwise, if $x \in T$ does *not* lie in such an F', denote by $I_1(x)$ the *smallest* interval I in the connected tree $\mathcal{F} \cap \mathcal{C}_T$ that contains x, and denote by $I_2(x)$ the *largest* interval in the connected tree $\mathcal{F} \cap \mathcal{C}_T$ that contains x. Then if $\widetilde{I_1}(x)$ denotes the child of $I_1(x)$ containing x, it is not one of the $F' \in \mathcal{K}_f$, and so we have

$$\begin{aligned}\left|\mathsf{P}_{\mathcal{C}_T}^\sigma f(x)\right| &= \left|\sum_{I \in \mathcal{F} \cap \mathcal{C}_T} \triangle_I^\sigma f(x)\right| = \left|\mathbb{E}_{\widetilde{I_1}(x)}^\sigma f(x) - \mathbb{E}_{I_2(x)}^\sigma f(x)\right| \\ &\leq \mathbb{E}_{\widetilde{I_1}(x)}^\sigma |f| + \mathbb{E}_{I_2(x)}^\sigma |f| \leq 2.\end{aligned}$$

We conclude that $\frac{1}{2}\mathsf{P}_{\mathcal{C}_T}^\sigma f \in \mathcal{MBF}_\sigma^{\left(\frac{\gamma}{2}\right)}(T)$ or $(L_T^\infty)_1(\sigma)$. The final assertion for $f \in \mathcal{MBF}_\sigma^{(\gamma)}(F)$ follows from the inequality $\mathbb{E}_{\theta K'}^\sigma \triangle_{\pi K'}^\sigma f < 0$, $K' \in \mathcal{K}_f$, itself a consequence of $\mathbb{E}_{K'}^\sigma \triangle_{\pi K'}^\sigma f > \frac{\gamma - 1}{2}$ and the fact that $\triangle_{\pi K'}^\sigma f$ has mean zero. □

One final observation is in store here, namely that we can always assume γ is as large as we wish in $\mathcal{RBF}_\sigma^{(\gamma)}(T)$ at the expense of dividing by a constant C_γ.

Lemma 5.18. *We have $\mathcal{RBF}_\sigma^{(\eta)}(T) \subset \frac{1}{4(\gamma+1)}\mathcal{RBF}_\sigma^{(\gamma)}(T)$ for $0 < \eta < \gamma < \infty$.*

Proof. Let $f \in \mathcal{RBF}_\sigma^{(\eta)}(T)$ with $f = g + h$; $g \in (L_T^\infty)_1(\sigma)$ and $h \in \mathcal{MBF}_\sigma^{(\eta)}(T)$. Then set

$$h_{bdd} = \sum_{K' \in \mathcal{K}_h :\ \mathbb{E}_{K'}^\sigma g \leq 2\gamma} \triangle_{\pi K'}^\sigma h \text{ and } h_{fluc} = \sum_{K' \in \mathcal{K}_h :\ \mathbb{E}_{K'}^\sigma g > 2\gamma} \triangle_{\pi K'}^\sigma h,$$

to obtain $\frac{1}{2(\gamma+1)}h_{bdd} \in (L^\infty_T)_1(\sigma)$ and $\frac{1}{2}h_{fluc} \in \mathcal{MBF}^{(\gamma)}_\sigma(T)$. Then $\frac{1}{4(\gamma+1)}f$ equals $\frac{1}{4(\gamma+1)}(g+h_{bdd})$ plus $\frac{1}{4(\gamma+1)}h_{fluc}$, which is in $\mathcal{RBF}^{(\gamma)}_\sigma(T)$. □

5.3. **The energy corona and stopping form.** In order to proceed with *interval* size splitting we must first impose an energy corona decomposition as in [NTV4] and [LaSaUr]. Recall the energy $\mathsf{E}(I,\omega)$ of a measure ω on a dyadic interval I is given by

$$\mathsf{E}(I,\omega)^2 = \frac{1}{|I|_\omega}\int_I \left(\frac{x-\mathbb{E}^\omega_I x}{|I|}\right)^2 d\omega(x) = \frac{1}{|I|_\omega}\sum_{J\subset I}\left|\left\langle \frac{x}{|I|}, h^\omega_J\right\rangle_\omega\right|^2,$$

where the second equality follows from the fact that the Haar functions $\{h^\omega_J\}_{J\subset I}$ form an orthonormal basis of $\left\{f\in L^2(\omega) : \mathrm{supp} f\subset I \text{ and } \int f d\omega = 0\right\}$. Recall also that $J \Subset I$ means $J\subset I$, $|J|\leq 2^{-r}|I|$ and that J is good - see Remark 3.26.

Definition 5.19. *Given an interval S_0, define $\mathcal{S}(S_0)$ to be the maximal subintervals $I\subset S_0$ such that there is a partition $\mathcal{J}(I)$ of I into good subintervals $J\Subset I$ with*

$$\sum_{J\in\mathcal{J}(I)} |J|_\omega \mathsf{E}(J,\omega)^2 \mathsf{P}(J,\mathbf{1}_{S_0}\sigma)^2 \geq 10\mathfrak{E}^2 \ |I|_\sigma, \tag{5.20}$$

where $\mathfrak{E}$ is the constant in the energy condition

$$\sum_{I\supset \dot{\bigcup}_i I_i} |I_i|_\omega \mathsf{E}(I_i,\omega)^2 \mathsf{P}(I_i,\mathbf{1}_I\sigma)^2 \leq \mathfrak{E}^2 \ |I|_\sigma .$$

Then define the σ-energy stopping intervals of S_0 to be the collection $\mathcal{S} = \bigcup_{n=0}^\infty \mathcal{S}_n$ where $\mathcal{S}_0 = \mathcal{S}(S_0)$ and $\mathcal{S}_{n+1} = \bigcup_{S\in\mathcal{S}_n}\mathcal{S}(S)$ for $n\geq 0$.

From the energy condition we obtain the σ-Carleson estimate

$$\sum_{S\in\mathcal{S}:\ S\subset I} |S|_\sigma \leq 2|I|_\sigma, \qquad I\in\mathcal{D}^\sigma. \tag{5.21}$$

We emphasize that this collection of stopping times depends only on S_0 and the weight pair (σ,ω), and not on any functions at hand. There is also a dual definition of energy stopping times $\mathcal{T}$ that satisfies an ω-Carleson estimate

$$\sum_{T\in\mathcal{T}:\ T\subset J} |T|_\omega \leq 2|J|_\omega, \qquad J\in\mathcal{D}^\omega. \tag{5.22}$$

Finally, we record the reason for introducing energy stopping times. If

$$X(\mathcal{C}_S)^2 \equiv \sup_{I\in\mathcal{C}_S}\frac{1}{|I|_\sigma}\sup_{\text{partitions } \mathcal{J}(I) \text{ of } I}\sum_{J\in\mathcal{J}(I)} |J|_\omega \mathsf{E}(J,\omega)^2 \mathsf{P}(J,\mathbf{1}_S\sigma)^2 \tag{5.23}$$

is (the square of) the *stopping energy* of the weight pair (σ,ω) with respect to the corona $\mathcal{C}_S$, then we have the *stopping energy bounds*

$$X(\mathcal{C}_S) \leq \sqrt{10}\mathfrak{E}, \qquad S\in\mathcal{S}. \tag{5.24}$$

Later we will introduce refinements of the stopping energy that depend as well on the Haar supports of the functions $f\in L^2(\sigma)$ and $g\in L^2(\omega)$ at hand.

5.4. **The parallel triple corona decomposition.** Here is our triple corona decomposition of $f\in L^{2}(\sigma)$. We first apply the Calderón-Zygmund corona decomposition to the function $f\in L^{2}(\sigma)$ obtain

$$f=\sum_{F\in\mathcal{F}}\mathsf{P}_{\mathcal{C}_{F}^{\sigma}}^{\sigma}f.$$

Then we apply part (1) of Proposition 5.8 to write

$$\mathsf{P}_{\mathcal{C}_{F}^{\sigma}}^{\sigma}f=\left(\mathsf{P}_{\mathcal{C}_{F}^{\sigma}}^{\sigma}f\right)_{bdd}+\left(\mathsf{P}_{\mathcal{C}_{F}^{\sigma}}^{\sigma}f\right)_{fluc}^{+}+\left(\mathsf{P}_{\mathcal{C}_{F}^{\sigma}}^{\sigma}f\right)_{fluc}^{-},$$

where $\frac{1}{\mathbb{E}_{F}^{\sigma}|f|}\left(\mathsf{P}_{\mathcal{C}_{F}^{\sigma}}^{\sigma}f\right)_{bdd}$ is a prebounded function on F and $\frac{1}{\mathbb{E}_{F}^{\sigma}|f|}\left(\mathsf{P}_{\mathcal{C}_{F}^{\sigma}}^{\sigma}f\right)_{fluc}^{\pm}$ is a prefluctuation$^{\pm}$ function on F. So as not to further clutter notation we will drop this distinction, and simply write $\mathsf{P}_{\mathcal{C}_{F}^{\sigma}}^{\sigma}f$ with the understanding that $\mathsf{P}_{\mathcal{C}_{F}^{\sigma}}^{\sigma}f$ represents $\mathbb{E}_{F}^{\sigma}|f|$ times either a prebounded or prefluctuation function on F.

We then iterate with a second Calderón-Zygmund corona decomposition as in part (2) of Proposition 5.8, and use Lemma 5.18 to ensure that the minimal bounded fluctuation functions have γ large. Lemma 1.12 on iterating coronas then gives us stopping times $\mathcal{K}=\mathcal{K}(\mathcal{F})$ and stopping data $\alpha_{\mathcal{K}(\mathcal{F})}(K)$ for f, along with the *double corona decomposition*

$$f=\sum_{F\in\mathcal{F}}\mathsf{P}_{\mathcal{C}_{F}^{\sigma}}^{\sigma}f=\sum_{K\in\mathcal{K}(\mathcal{F})}\mathsf{P}_{\mathcal{C}_{K}^{\sigma}}^{\sigma}f. \tag{5.25}$$

Keeping in mind our understanding regarding $\mathsf{P}_{\mathcal{C}_{F}^{\sigma}}^{\sigma}f$ above, we have the following estimate for $\mathsf{P}_{\mathcal{C}_{K}^{\sigma}}^{\sigma}f$ where we define $F_K\in\mathcal{F}$ to be the unique stopping interval in $\mathcal{F}$ for which $K\in\mathcal{C}_{F}^{\sigma}$ (recall we have arranged for γ to be large at the expense of increasing the constant C_γ below).

Lemma 5.26. *For $K\in\mathcal{K}(\mathcal{F})$ and F_K such that $K\in\mathcal{C}_{F_K}^{\sigma}$, we have*

$$\frac{1}{C_{\gamma}\left(\mathbb{E}_{K}^{\sigma}\left|\mathsf{P}_{\mathcal{C}_{F_K}^{\sigma}}^{\sigma}f\right|\right)}\mathsf{P}_{\mathcal{C}_{K}^{\sigma}}^{\sigma}f\in\mathcal{RBF}_{\sigma}^{(\gamma)}(K).$$

Proof. Let $F=F_K$. By Lemma 5.2, we have

$$h\equiv\frac{1}{(C_0\gamma+\gamma+1)\,\mathbb{E}_{F}^{\sigma}|f|}\mathsf{P}_{\mathcal{C}_{F}^{\sigma}}^{\sigma}f\in\mathcal{BF}_{\sigma}^{(\gamma)}(F), \tag{5.27}$$

and then by Proposition 5.8 and Lemma 5.18, we conclude

$$\begin{aligned}\frac{1}{5\mathbb{E}_{K}^{\sigma}\left|\mathsf{P}_{\mathcal{C}_{F}^{\sigma}}^{\sigma}f\right|}\mathsf{P}_{\mathcal{C}_{K}^{\sigma}}^{\sigma}\mathsf{P}_{\mathcal{C}_{F}^{\sigma}}^{\sigma}f &= \frac{1}{5\mathbb{E}_{K}^{\sigma}|h|}\mathsf{P}_{\mathcal{C}_{K}^{\sigma}}^{\sigma}h\\ &\in \mathcal{RBF}_{\sigma}^{\left(\frac{1}{2}\right)}(K)\subset 4(\gamma+1)\,\mathcal{RBF}_{\sigma}^{(\gamma)}(K).\end{aligned}$$

□

We then finish our triple corona decomposition of f in (5.25) as follows. For each fixed $K\in\mathcal{K}(\mathcal{F})$, construct the *energy* corona decomposition $\{\mathcal{C}_{S}^{\sigma}\}_{S\in\mathcal{S}(K)}$ corresponding to the weight pair (σ,ω) with top interval $S_0=K$, as given in Definition 5.19. Recall from Lemma 5.26 that

$$h_K\equiv\frac{1}{C_{\gamma}\left(\mathbb{E}_{K}^{\sigma}\left|\mathsf{P}_{\mathcal{C}_{F_K}^{\sigma}}^{\sigma}f\right|\right)}\mathsf{P}_{\mathcal{C}_{K}^{\sigma}}^{\sigma}f\in\mathcal{RBF}_{\sigma}^{(\gamma)}(K).$$

We now modify $\mathcal{S}(K)$ by adding the intervals $K' \in \mathcal{K}_{h_K}$ to $\mathcal{S}(K)$ and removing from $\mathcal{S}(K)$ all the intervals S that are strictly contained in some $K' \in \mathcal{K}_{h_K}$. We denote this modified collection by $\mathcal{S}'(K)$. Of course, if $h_K \in (L^\infty_K)_1(\sigma)$, then $\mathcal{K}_{h_K} = \emptyset$ and no modification is made, so that $\mathcal{S}'(K) = \mathcal{S}(K)$.

We then define stopping data $\left\{\alpha_{\mathcal{S}'(K)}(S)\right\}_{S\in\mathcal{S}'(K)}$ for the function $\mathsf{P}_{\mathcal{C}^\sigma_K} f$ relative to the modified stopping times $\mathcal{S}'(K)$ as follows. For $K \in \mathcal{K}(\mathcal{F})$ define

$$\alpha_{\mathcal{S}'(K)}(S) = \begin{cases} 2\alpha_{\mathcal{K}}(K) & \text{for} \quad S \in \mathcal{S}'(K) \setminus \mathcal{K}_{h_K} \\ \alpha_{\mathcal{K}}(K') & \text{for} \quad S \in \mathcal{K}_{h_K} \end{cases}.$$

Then properties (2) and (4) of Definition 1.8 are immediate. Property (1) follows since if $I \in \mathcal{C}^{\mathcal{K}}_K$, then

$$\mathbb{E}^\sigma_I \left|\mathsf{P}_{\mathcal{C}^{\mathcal{K}}_K} f\right| \leq 2\mathbb{E}^\sigma_I \left|\mathsf{P}_{\mathcal{C}^{\mathcal{F}}_{F_K}} f\right| \leq 2\alpha_{\mathcal{F}}(F_K) \leq 2\alpha_{\mathcal{K}}(K).$$

Property (3) follows because (5.21) gives

$$\begin{aligned} \sum_{S\in\mathcal{S}'(K)} \alpha_{\mathcal{S}'(K)}(S)^2 |S|_\sigma &\lesssim \int \left(\sum_{S\in\mathcal{S}'(K)\setminus\mathcal{K}_{h_K}} \alpha_{\mathcal{S}'(K)}(S)\mathbf{1}_S\right)^2 d\sigma + \int \left(\sum_{S\in\mathcal{K}_{h_K}} \alpha_{\mathcal{S}'(K)}(S)\mathbf{1}_S\right)^2 d\sigma \\ &= \sum_{S,S'\in\mathcal{S}(K)\setminus\mathcal{K}_{h_K}} 4\alpha_{\mathcal{K}}(K)^2 |S\cap S'|_\sigma + \sum_{K'\in\mathcal{K}_{h_K}} \alpha_{\mathcal{K}}(K')^2 |K'|_\sigma \\ &\leq 8\alpha_{\mathcal{K}}(K)^2 \sum_{S\in\mathcal{S}(K)} \sum_{\substack{S'\in\mathcal{S}(K)\\ S'\subset S}} |S\cap S'|_\sigma + \sum_{K'\in\mathcal{K}_{h_K}} \alpha_{\mathcal{K}}(K')^2 |K'|_\sigma \\ &\leq 16\alpha_{\mathcal{K}}(K)^2 \sum_{S\in\mathcal{S}(K)} |S|_\sigma + \sum_{K'\in\mathcal{K}_{h_K}} \alpha_{\mathcal{K}}(K')^2 |K'|_\sigma \\ &\leq 16\alpha_{\mathcal{K}}(K)^2 |K|_\sigma + \sum_{K'\in\mathcal{K}_{h_K}} \alpha_{\mathcal{K}}(K')^2 |K'|_\sigma \lesssim \left\|\mathsf{P}_{\mathcal{C}^{\mathcal{K}}_K} f\right\|^2_{L^2(\sigma)}. \end{aligned}$$

At this point we write $\mathcal{S}(K)$ in place of $\mathcal{S}'(K)$ and apply Lemma 1.12 to obtain iterated stopping times $\mathcal{S}(\mathcal{K}(\mathcal{F}))$ and iterated stopping data $\left\{\alpha_{\mathcal{S}(\mathcal{K}(\mathcal{F}))}(S)\right\}_{S\in\mathcal{S}(\mathcal{K}(\mathcal{F}))}$. This gives us the following *triple corona decomposition* of f,

$$\begin{aligned} f &= \sum_{F\in\mathcal{F}} \mathsf{P}^\sigma_{\mathcal{C}^\sigma_F} f = \sum_{F\in\mathcal{F}} \sum_{K\in\mathcal{K}(F)} \mathsf{P}^\sigma_{\mathcal{C}^\sigma_K} \mathsf{P}^\sigma_{\mathcal{C}^\sigma_F} f = \sum_{F\in\mathcal{F}} \sum_{K\in\mathcal{K}(F)} \sum_{S\in\mathcal{S}(K)} \mathsf{P}^\sigma_{\mathcal{C}^\sigma_S} \mathsf{P}^\sigma_{\mathcal{C}^\sigma_K} \mathsf{P}^\sigma_{\mathcal{C}^\sigma_F} f \\ &= \sum_{K\in\mathcal{K}(\mathcal{F})} \sum_{S\in\mathcal{S}(K)} \mathsf{P}^\sigma_{\mathcal{C}^\sigma_S\cap\mathcal{C}^\sigma_K} f = \sum_{S\in\mathcal{S}(\mathcal{K}(\mathcal{F}))} \mathsf{P}^\sigma_{\mathcal{C}^\sigma_S} f, \end{aligned} \tag{5.28}$$

as well as a corresponding triple corona decomposition of g,

$$\begin{aligned} g &= \sum_{G\in\mathcal{G}} \mathsf{P}^\omega_{\mathcal{C}^\omega_G} g = \sum_{G\in\mathcal{G}} \sum_{L\in\mathcal{L}(G)} \mathsf{P}^\omega_{\mathcal{C}^\omega_L} \mathsf{P}^\omega_{\mathcal{C}^\omega_G} g = \sum_{G\in\mathcal{G}} \sum_{L\in\mathcal{L}(G)} \sum_{T\in\mathcal{T}(L)} \mathsf{P}^\omega_{\mathcal{C}^\omega_T} \mathsf{P}^\omega_{\mathcal{C}^\omega_L} \mathsf{P}^\omega_{\mathcal{C}^\omega_G} g \\ &= \sum_{L\in\mathcal{L}(\mathcal{G})} \sum_{T\in\mathcal{T}(L)} \mathsf{P}^\omega_{\mathcal{C}^\omega_T\cap\mathcal{C}^\omega_L} g = \sum_{T\in\mathcal{T}(\mathcal{L}(\mathcal{G}))} \mathsf{P}^\omega_{\mathcal{C}^\omega_T} g. \end{aligned} \tag{5.29}$$

We emphasize that the energy coronas $\mathcal{S}$ and $\mathcal{T}$ are independent of each other, in contrast to the usual constructions in [NTV4] and [LaSaUr], where $\mathcal{T}$ is derived from $\mathcal{S}$. Using Lemma 5.17, we have the following extension of Lemma 5.26.

Lemma 5.30. *For $S\in\mathcal{S}(\mathcal{K}(\mathcal{F}))$ and $T\in\mathcal{T}(\mathcal{L}(\mathcal{G}))$, and with corresponding K,F and L,G as above, we have*

$$\begin{aligned}\frac{1}{C_\gamma\left(\mathbb{E}_K^\sigma\left|\mathsf{P}_{\mathcal{C}_{F_K}^\sigma}^\sigma f\right|\right)}\mathsf{P}_{\mathcal{C}_S^\sigma\cap\mathcal{C}_K^\sigma}^\sigma f &\in \mathcal{RBF}_\sigma^{(\gamma)}(S),\\ \frac{1}{C_\gamma\left(\mathbb{E}_L^\omega\left|\mathsf{P}_{\mathcal{C}_{G_L}^\omega}^\omega g\right|\right)}\mathsf{P}_{\mathcal{C}_T^\omega\cap\mathcal{C}_L^\omega}^\omega g &\in \mathcal{RBF}_\omega^{(\gamma)}(T).\end{aligned}$$

Now we apply the *parallel* corona decomposition as in (1.7) corresponding to the triple corona decompositions (5.28) and (5.29). We obtain

$$\begin{aligned}\langle H_\sigma f,g\rangle_\omega &= \sum_{F\in\mathcal{F}}\sum_{K\in\mathcal{K}}\sum_{S\in\mathcal{S}}\sum_{G\in\mathcal{G}}\sum_{L\in\mathcal{L}}\sum_{T\in\mathcal{T}}\left\langle H_\sigma\left(\mathsf{P}_{\mathcal{C}_S^\sigma}^\sigma\mathsf{P}_{\mathcal{C}_K^\sigma}^\sigma\mathsf{P}_{\mathcal{C}_F^\sigma}^\sigma f\right),\mathsf{P}_{\mathcal{C}_T^\omega}^\omega\mathsf{P}_{\mathcal{C}_L^\omega}^\omega\mathsf{P}_{\mathcal{C}_G^\omega}^\omega g\right\rangle_\omega\\ &= \sum_{K\in\mathcal{K}(\mathcal{F})}\sum_{S\in\mathcal{S}}\sum_{L\in\mathcal{L}(\mathcal{G})}\sum_{T\in\mathcal{T}}\left\langle H_\sigma\mathsf{P}_{\mathcal{C}_S^\sigma\cap\mathcal{C}_K^\sigma}^\sigma f,\mathsf{P}_{\mathcal{C}_T^\omega\cap\mathcal{C}_L^\omega}^\omega g\right\rangle_\omega\\ &\equiv \sum_{A\in\mathcal{A}}\sum_{B\in\mathcal{B}}\left\langle H_\sigma\left(\mathsf{P}_{\mathcal{C}_A^\sigma}^\sigma f\right),\mathsf{P}_{\mathcal{C}_B^\omega}^\omega g\right\rangle_\omega\\ &= \mathsf{H}_{near}(f,g)+\mathsf{H}_{disjoint}(f,g)+\mathsf{H}_{far}(f,g),\end{aligned}$$

where

$$\mathcal{A}\equiv\mathcal{S}(\mathcal{K}(\mathcal{F}))\text{ and }\mathcal{B}\equiv\mathcal{T}(\mathcal{L}(\mathcal{G}))$$

are the triple stopping collections for f and g respectively. We are relabeling the triple coronas as $\mathcal{A}$ and $\mathcal{B}$ here so as to minimize confusion when we apply the various different estimates associated with each of the three corona decompositions of f and g. We now record the two main facts proved above.

Lemma 5.31. *The data $\mathcal{A}$ and $\{\alpha_{\mathcal{A}}(A)\}_{A\in\mathcal{A}}$ satisfy properties (1), (2), (3) and (4) in Definition 1.8, and similarly for the data $\mathcal{B}$ and $\{\beta_{\mathcal{B}}(B)\}_{B\in\mathcal{B}}$. Moreover, we have the estimates*

$$\begin{aligned}\frac{1}{C_\gamma\alpha_{\mathcal{A}}(A)}\mathsf{P}_{\mathcal{C}_A^\sigma}^\sigma f &\in \mathcal{RBF}_\sigma^{(\gamma)}(A),\\ \frac{1}{C_\gamma\beta_{\mathcal{B}}(B)}\mathsf{P}_{\mathcal{C}_B^\omega}^\omega g &\in \mathcal{RBF}_\omega^{(\gamma)}(B),\end{aligned}$$

where the constant C_γ depends only on $\gamma>0$, which can be taken as large as we wish.

Thus we can apply the Iterated Corona Proposition 1.11 to the parallel triple corona decomposition (1.7):

$$\langle H_\sigma f,g\rangle_\omega=\mathsf{H}_{near}(f,g)+\mathsf{H}_{disjoint}(f,g)+\mathsf{H}_{far}(f,g).$$

The result is that

$$|\mathsf{H}_{far}(f,g)|\lesssim(\mathfrak{NTV})\,\|f\|_{L^2(\sigma)}\,\|g\|_{L^2(\omega)}.$$

Moreover, Lemma 3.1 implies

$$|\mathsf{H}_{disjoint}(f,g)|\lesssim(\mathfrak{NTV})\,\|f\|_{L^2(\sigma)}\,\|g\|_{L^2(\omega)},$$

and so it remains to deal only with the near form $\mathsf{H}_{near}(f,g)$.

We first further decompose $\mathsf{H}_{near}(f,g)$ into lower and upper parts:

$$\begin{aligned}
\mathsf{H}_{near}(f,g) &= \left\{ \sum_{\substack{(A,B)\in\mathsf{Near}(\mathcal{A}\times\mathcal{B})\\ B\subset A}} + \sum_{\substack{(A,B)\in\mathsf{Near}(\mathcal{A}\times\mathcal{B})\\ A\subset B}} \right\} \int H_\sigma\left(\mathsf{P}^\sigma_{\mathcal{C}_A} f\right) \left(\mathsf{P}^\omega_{\mathcal{C}_B} g\right)\omega \\
&= \mathsf{H}_{near\ lower}(f,g) + \mathsf{H}_{near\ upper}(f,g); \\
\mathsf{H}_{near\ lower}(f,g) &= \sum_{A\in\mathcal{A}} \left\langle H_\sigma \mathsf{P}^\sigma_{\mathcal{C}^\sigma_A} f, \mathsf{Q}^\omega_{\widetilde{\mathcal{C}}^\omega_A} g\right\rangle_\omega; \\
\mathsf{Q}^\omega_{\widetilde{\mathcal{C}}^\omega_A} &\equiv \sum_{J\in\widetilde{\mathcal{C}}_A} \triangle^\omega_J \text{ where } \widetilde{\mathcal{C}}^\omega_A \equiv \bigcup_{\substack{B\in\mathcal{B}:\ B\subset A\\ (A,B)\in\mathsf{Near}(\mathcal{A}\times\mathcal{B})}} \mathcal{C}^\omega_B.
\end{aligned}$$

Thus we have that $\mathsf{Q}^\omega_{\widetilde{\mathcal{C}}_A} = \sum_{\substack{B\in\mathcal{B}:\ B\subset A\\ (A,B)\in\mathsf{Near}(\mathcal{A}\times\mathcal{B})}} \mathsf{P}^\omega_{\mathcal{C}_B}$ is the projection onto all of the coronas $\mathcal{C}^\omega_B$ for which B is 'near and below' A. By symmetry, it suffices to consider the lower near form $\mathsf{H}_{near\ lower}(f,g)$.

5.4.1. *Reduction to restricted bounded fluctuation.* By Lemma 5.17, the function $\mathsf{P}^\sigma_{\mathcal{C}^\sigma_A} f$ is an appropriate multiple of a function in $\mathcal{RBF}^{(\gamma)}_\sigma(A)$. More precisely, if $A = S \in \mathcal{S}(\mathcal{K}(\mathcal{F}))$ and K_S is the unique interval $K \in \mathcal{K}(\mathcal{F})$ satisfying $S \in \mathcal{S}(K)$, then $\mathsf{P}^\sigma_{\mathcal{C}^\sigma_A} f = \mathsf{P}^\sigma_{\mathcal{C}^\sigma_S \cap \mathcal{C}^\sigma_{K_S}} f$ and

$$f_S \equiv \frac{1}{C_\gamma\left(\mathbb{E}^\sigma_{K_S}\left|\mathsf{P}^\sigma_{\mathcal{C}^\sigma_{F_{K_S}}} f\right|\right)} \mathsf{P}^\sigma_{\mathcal{C}^\sigma_S\cap\mathcal{C}^\sigma_{K_S}} f \in \mathcal{RBF}^{(\gamma)}_\sigma(S).$$

By the definition of $\mathcal{RBF}^{(\gamma)}_\sigma(S)$, we can write

$$\mathsf{P}^\sigma_{\mathcal{C}^\sigma_S\cap\mathcal{C}^\sigma_{K_S}} f = \varphi_S + \psi_S$$

where

$$\frac{1}{C_\gamma\left(\mathbb{E}^\sigma_K\left|\mathsf{P}^\sigma_{\mathcal{C}^\sigma_{F_{K_S}}} f\right|\right)} \varphi_S \in \left(L^\infty_S\right)_1(\sigma), \tag{5.32}$$

and

$$\frac{1}{C_\gamma\left(\mathbb{E}^\sigma_K\left|\mathsf{P}^\sigma_{\mathcal{C}^\sigma_{F_{K_S}}} f\right|\right)} \psi_S \in \mathcal{MBF}^{(\gamma)}_\sigma(S). \tag{5.33}$$

We now apply, and **for the only time in this paper**, the first of the indicator/interval testing conditions in (1.3) to obtain

$$\left|\left\langle H_\sigma\varphi_S, \mathsf{Q}^\omega_{\widetilde{\mathcal{C}}^\omega_S} g\right\rangle_\omega\right| \le \mathfrak{T}_{ind} 5\left(\mathbb{E}^\sigma_K\left|\mathsf{P}^\sigma_{\mathcal{C}^\sigma_{F_{K_S}}} f\right|\right)\sqrt{|S|_\sigma}\left\|\mathsf{Q}^\omega_{\widetilde{\mathcal{C}}^\omega_S} g\right\|_{L^2(\omega)}.$$

If we can also show that

(5.34)
$$\left|\left\langle H_\sigma\psi_S, \mathsf{Q}^\omega_{\widetilde{\mathcal{C}}^\omega_S} g\right\rangle_\omega\right| \lesssim \mathfrak{NTV}\left\{\|\psi_S\|_{L^2(\sigma)} + 5\left(\mathbb{E}^\sigma_K\left|\mathsf{P}^\sigma_{\mathcal{C}^\sigma_{F_{K_S}}} f\right|\right)\sqrt{|S|_\sigma}\right\}\left\|\mathsf{Q}^\omega_{\widetilde{\mathcal{C}}^\omega_S} g\right\|_{L^2(\omega)},$$

it then follows that

$$\left|\left\langle H_\sigma \mathsf{P}^\sigma_{\mathcal{C}^\sigma_A} f, \mathsf{Q}^\omega_{\widetilde{\mathcal{C}}^\omega_A} g\right\rangle_\omega\right| = \left|\left\langle H_\sigma \mathsf{P}^\sigma_{\mathcal{C}^\sigma_S \cap \mathcal{C}^\sigma_{K_S}} f, \mathsf{Q}^\omega_{\widetilde{\mathcal{C}}^\omega_A} g\right\rangle_\omega\right|$$

$$\lesssim \mathfrak{NTV}_{ind} \left\{\left\|\mathsf{P}^\sigma_{\mathcal{C}^\sigma_S \cap \mathcal{C}^\sigma_{K_S}}\right\|_{L^2(\sigma)} + 5\left(\mathbb{E}^\sigma_K \left|\mathsf{P}^\sigma_{\mathcal{C}^\sigma_{F_{K_S}}} f\right|\right)\sqrt{|S|_\sigma}\right\} \left\|\mathsf{Q}^\omega_{\widetilde{\mathcal{C}}^\omega_S} g\right\|_{L^2(\omega)}$$

$$\approx \mathfrak{NTV}_{ind} \left\{\left\|\mathsf{P}^\sigma_{\mathcal{C}^\sigma_A} f\right\|_{L^2(\sigma)} + \alpha_{\mathcal{A}}(A)\sqrt{|A|_\sigma}\right\} \left\|\mathsf{Q}^\omega_{\widetilde{\mathcal{C}}^\omega_A} g\right\|_{L^2(\omega)},$$

and hence that

$$|\mathsf{H}_{near\ lower}(f,g)| \leq \sum_{A\in\mathcal{A}} \left|\left\langle H_\sigma \mathsf{P}^\sigma_{\mathcal{C}^\sigma_A} f, \mathsf{Q}^\omega_{\widetilde{\mathcal{C}}^\omega_A} g\right\rangle_\omega\right|$$

$$\lesssim \mathfrak{NTV}_{ind} \left(\sum_{A\in\mathcal{A}} \left\{\left\|\mathsf{P}^\sigma_{\mathcal{C}^\sigma_A} f\right\|^2_{L^2(\sigma)} + \alpha_{\mathcal{A}}(A)^2 |A|_\sigma\right\}\right)^{\frac{1}{2}} \left(\sum_{A\in\mathcal{A}} \left\|\mathsf{Q}^\omega_{\widetilde{\mathcal{C}}^\omega_A} g\right\|^2_{L^2(\omega)}\right)^{\frac{1}{2}}$$

$$\lesssim \mathfrak{NTV}_{ind} \|f\|_{L^2(\sigma)} \|g\|_{L^2(\omega)}.$$

Thus we have proved the following reduction of the two weight inequality for the Hilbert transform to testing restricted bounded fluctuation functions in (5.34).

In order to state the inequality precisely, we need two definitions. First, we introduce a refinement of the stopping energy in (5.23) that depends as well on functions f and g.

Definition 5.35. *Given $g \in L^2(\omega)$, define the g-energy $\mathsf{E}_g(J,\omega)$ of an interval J by*

$$\mathsf{E}_g(J,\omega)^2 \equiv \mathsf{E}_{\mathcal{G}}(J,\omega)^2 = \frac{1}{|J|_\omega} \sum_{J'\in\mathcal{G}} \left|\left\langle \frac{x}{|J|}, h^\omega_{J'}\right\rangle_\omega\right|^2,$$

where $\mathcal{G}$ is the Haar support of g and $\mathsf{E}_{\mathcal{G}}$ is defined in (3.7). For an interval I, let $\mathcal{J}_g(I)$ consist of the maximal *intervals J in $\mathcal{G}$ that satisfy $J \Subset I$. Then given $f \in L^2_S(\sigma)$ and $g \in L^2_S(\omega)$, define the* stopping energy *$\boldsymbol{X}^S(f,g)$ of the pair (f,g) on S by*

$$\boldsymbol{X}^S(f,g)^2 = \sup_{I\in\mathcal{F}} \frac{1}{|I|_\sigma} \sum_{J\in\mathcal{J}_g(I)} |J|_\omega \mathsf{E}_g(J,\omega)^2 \mathsf{P}\left(J, \mathbf{1}_{S\setminus I}\sigma\right)^2. \tag{5.36}$$

Second, we introduce a subspace $\widetilde{L}^2_A(\omega)$ of $L^2_A(\omega)$ that has a small amount of structure relative to the interval A, and which will play a role in reducing to stopping forms below.

Definition 5.37. *Define $g \in \widetilde{L}^2_A(\omega)$ if*

$$g = \mathsf{Q}^\omega_{\mathcal{C}^\omega_{\widetilde{A}}} g = \sum_{B\in\mathcal{B}:\ B\sim A} \mathsf{P}^\omega_{\mathcal{C}^\omega_B} g, \quad \mathcal{C}^\omega_{\widetilde{A}} \equiv \bigcup_{B\in\mathcal{B}:\ B\sim A} \mathcal{C}^\omega_B,$$

where the coronas $\{\mathcal{C}^\omega_B\}_{B\in\mathcal{B}:\ B\sim A}$ are as above, satisfy an ω-Carleson condition, and

$$\frac{1}{C_\gamma\left(\mathbb{E}^\omega_{L_T}\left|\mathsf{P}^\omega_{\mathcal{C}^\omega_{G_{L_T}}} g\right|\right)} \mathsf{P}^\omega_{\mathcal{C}^\omega_T \cap \mathcal{C}^\omega_{L_T}} g \in \mathcal{RBF}^{(\gamma)}_\omega(T),$$

where $B = T$, $\mathsf{P}^\omega_{\mathcal{C}^\omega_B} g = \mathsf{P}^\omega_{\mathcal{C}^\omega_T \cap \mathcal{C}^\omega_{L_T}} g$.

Here is the reduction we have proved above.

Lemma 5.38. *The two weight Hilbert transform inequality (1.1) is implied by the following minimal bilinear inequality with best constant $\mathfrak{M}$, and its dual inequality with best constant $\mathfrak{M}^*$:*

$$\left|\left\langle H_\sigma f,g\right\rangle_\omega\right| \lesssim \mathfrak{M}\left(\left\|f\right\|_{L^2(\sigma)}+\sqrt{\left|A\right|_\sigma}\right)\left\|g\right\|_{L^2(\omega)}, \tag{5.39}$$

$$\text{for } f\in\mathcal{MBF}_\sigma^{(\gamma)}(A),\ g\in\widetilde{L}_A^2(\omega) \text{ and } \mathbf{X}^A(f,g)\leq\sqrt{10}\mathfrak{C}.$$

The occurrence of $f\in\mathcal{MBF}_\sigma^{(\gamma)}(A)$ in (5.39) as a minimal bounded fluctuation function is the best that can be hoped for regarding f. But we still have two problems with $g\in\widetilde{L}_A^2(\omega)$: first, that g is an *unbounded sum* of restricted bounded fluctuation functions; and second, that these summands are *not* minimal bounded fluctuation, just restricted.

6. Interval size splitting

It remains to estimate $\left\langle H_\sigma f,g\right\rangle_\omega$ in (5.39) for $f\in\mathcal{MBF}_\sigma^{(\gamma)}(A)$ and $g\in\widetilde{L}_A^2(\omega)$. For this we will finally resort to the original interval size splitting of Nazarov, Treil and Volberg. We will expand the functions f and g in their Haar decompositions over $I\in\mathcal{D}^\sigma$ and $J\in\mathcal{D}^\omega$ respectively, and then apply the NTV splitting according to the relative length of the intervals, $|J|<|I|$ or $|I|<|J|$. The key advantages we have that permit this splitting to work in the current situation are that

(1) The function f lies in $\mathcal{MBF}_\sigma^{(\gamma)}(A)$, and the function $g=\sum_{B\in\mathcal{B}:\ B\sim A}\mathsf{P}_{\mathcal{C}_B^\omega}^\omega g$ lies in $\widetilde{L}_A^2(\omega)$;

(2) We have a stopping energy bound,

$$\mathbf{X}(f,g)\leq\sqrt{10}\mathfrak{E}\equiv C_{\mathbf{X}}\lesssim\sqrt{\mathcal{A}_2}+\mathfrak{T}, \tag{6.1}$$

where $\mathbf{X}(f,g)$ is the *stopping energy* as defined in (5.36) below.

(3) There is also a dual stopping energy bound

$$\mathbf{X}'(f_B,g_B)\leq C_{\mathbf{X}}\lesssim\sqrt{\mathcal{A}_2}+\mathfrak{T}, \tag{6.2}$$

for the corona decomposition $\{\mathcal{C}_B^\omega\}_{B\in\mathcal{B}}$, where f_B,g_B are defined below.

The boundedness of the form

$$\mathsf{B}^A(f,g)\equiv\left\langle H_\sigma f,g\right\rangle_\omega$$

in (5.39) is implied by boundedness of each of the split forms $\mathsf{B}_\Subset(f,g)$ and $\mathsf{B}_\Supset(f,g)$ introduced in [LaSaShUr],

$$\begin{aligned}\mathsf{B}_\Subset^A(f,g) &= \sum_{\substack{(I,J)\in\left(\mathcal{C}_A^\sigma\cap\mathcal{C}_{\widetilde{A}}^\omega\right)\times\mathcal{C}_{\widetilde{A}}^\omega\\ J\Subset I}}\left\langle H_\sigma\bigtriangleup_I^\sigma f,\bigtriangleup_J^\omega g\right\rangle_\omega,\\ \mathsf{B}_\Supset^A(f,g) &= \sum_{\substack{(I,J)\in\left(\mathcal{C}_A^\sigma\cap\mathcal{C}_{\widetilde{A}}^\omega\right)\times\mathcal{C}_{\widetilde{A}}^\omega\\ I\Subset J}}\left\langle\bigtriangleup_I^\sigma f,H_\omega\bigtriangleup_J^\omega g\right\rangle_\sigma,\end{aligned}$$

where the presence of the superscript A in the forms $\mathsf{B}^A(f,g)$, $\mathsf{B}_\Subset^A(f,g)$ and $\mathsf{B}_\Supset^A(f,g)$ indicates that f and g are as in (5.39), and $\mathcal{C}_{\widetilde{A}}^\omega$ is defined in Definition 5.37.

Now the function 'on top' in the form $\mathsf{B}_\Subset^A(f,g)$, namely f, has the special property of belonging to $\mathcal{MBF}_\sigma(A)$. The function 'on bottom' in the form, namely g, lies in the broader space $\widetilde{L}_A^2(\omega)$, so in particular in $L_A^2(\omega)$, and the pair satisfies the stopping energy bound in (6.1). We say that such a form is of type $\mathcal{MBF}/L^2$, reflecting the fact that the top function is $\mathcal{MBF}$ and the bottom function is L^2. However, the 'top' function in the form $\mathsf{B}_{\Supset}^{A}(f,g)$, namely g, fails to be $\mathcal{MBF}$, rather it is a sum of such, and so $\mathsf{B}_{\Supset}^{A}(f,g)$ is *not* a form of type $\mathcal{MBF}/L^2$. Before we can proceed with an application of the NTV method, we must further reduce the boundedness of the form $\mathsf{B}_{\Supset}^{A}(f,g)$ to that of simpler forms. Recall that $g=\sum_{B\sim A}\mathsf{P}_{\mathcal{C}_B^{\omega}}^{\omega}g$, and write

$$\mathsf{B}_{\Supset}^{A}(f,g)=\sum_{B\sim A}\sum_{\substack{(I,J)\in\left(\mathcal{C}_A^{\sigma}\cap\mathcal{C}_{\widetilde{A}}^{\omega}\right)\times\mathcal{C}_{\widetilde{A}}^{\omega}\\ I\Subset J}}\left\langle\triangle_I^{\sigma}f,H_{\omega}\triangle_J^{\omega}\left(\mathsf{P}_{\mathcal{C}_B^{\omega}}^{\omega}g\right)\right\rangle_{\sigma}.$$

We claim that boundedness in (5.39), modulo $(\mathcal{NTV})\left\|f\right\|_{L^2(\sigma)}\left\|g\right\|_{L^2(\omega)}$, of the form $\mathsf{B}_{\Supset}^{A}$ is implied by boundedness in (5.39) of the local form

$$\begin{aligned}\mathsf{B}_{\Supset,\ local}^{A}(f,g)&\equiv\sum_{B\sim A}\alpha_{\mathcal{B}}(B)\,\mathsf{B}_{\Supset,B}^{A}(f_B,g_B);\\ \mathsf{B}_{\Supset,B}^{A}(f_B,g_B)&\equiv\sum_{\substack{(I,J)\in(\mathcal{C}_A^{\sigma}\cap\mathcal{C}_B^{\omega})\times\mathcal{C}_B^{\omega}\\ I\Subset J}}\left\langle\triangle_I^{\sigma}f_B,H_{\omega}\triangle_J^{\omega}g_B\right\rangle_{\sigma},\\ f_B&=\mathsf{P}_{\mathcal{C}_A^{\sigma}\cap\mathcal{C}_B^{\omega}}^{\sigma}f,\qquad g_B=\frac{1}{\alpha_{\mathcal{B}}(B)}\mathsf{P}_{\mathcal{C}_B^{\omega}}^{\omega}g.\end{aligned}$$

The key point here is that the difference of the forms in question is given by

$$\mathsf{B}_{\Supset}^{A}(f,g)-\mathsf{B}_{\Supset,\ local}^{A}(f,g)=\sum_{B\sim A}\sum_{\substack{(I,J)\in\left(\mathcal{C}_A^{\sigma}\cap\mathcal{C}_{\widetilde{A}}^{\omega}\right)\times\mathcal{C}_B^{\omega}\\ I\Subset B'\in\mathfrak{C}_{\mathcal{B}}(B)}}\left\langle\triangle_I^{\sigma}f,H_{\omega}\triangle_J^{\omega}g\right\rangle_{\sigma}=\mathsf{B}_{mix}(f,g),\tag{6.3}$$

and so the estimate for $\mathsf{B}_{mix}(f,g)$ in Proposition 4.2 applies to prove our claim. Altogether we have shown that (5.39) will follow from the two inequalities,

$$\begin{aligned}&\left|\mathsf{B}_{\Subset}^{A}(f,g)\right|\lesssim\mathfrak{M}\left(\left\|f\right\|_{L^2(\sigma)}+\sqrt{\left|A\right|_{\sigma}}\right)\left\|g\right\|_{L^2(\omega)},\\ &\text{for }f\in\mathcal{MBF}_{\sigma}^{(\gamma)}(A),\ g\in\widetilde{L}_A^2(\omega)\ \text{ and }\boldsymbol{X}^A(f,g)\leq\sqrt{10}\mathfrak{C},\end{aligned}$$

and

$$\begin{aligned}&\left|\sum_{B\sim A}\alpha_{\mathcal{B}}(B)\,\mathsf{B}_{\Supset,B}^{A}(f_B,g_B)\right|\lesssim\mathfrak{M}\left(\left\|f\right\|_{L^2(\sigma)}+\sqrt{\left|A\right|_{\sigma}}\right)\left\|g\right\|_{L^2(\omega)},\\ &\text{for }f\in\mathcal{MBF}_{\sigma}^{(\gamma)}(A),\ g\in\widetilde{L}_A^2(\omega)\ \text{ and }\boldsymbol{X}^A(f,g)\leq\sqrt{10}\mathfrak{C},\\ &f_B=\mathsf{P}_{\mathcal{C}_A^{\sigma}\cap\mathcal{C}_B^{\omega}}^{\sigma}f,\qquad g_B=\frac{1}{\alpha_{\mathcal{B}}(B)}\mathsf{P}_{\mathcal{C}_B^{\omega}}^{\omega}g.\end{aligned}$$

Moreover, the second inequality will follow from Cauchy-Schwarz and

$$\begin{aligned}&\left|\mathsf{B}_{\Supset,B}^{A}(f_B,g_B)\right|\lesssim\mathfrak{M}\left\|f_B\right\|_{L^2(\sigma)}\left(\left\|g_B\right\|_{L^2(\omega)}+\sqrt{\left|B\right|_{\omega}}\right),\\ f_B\ \in\ &\mathcal{MBF}_{\sigma}(B),\ g_B\in\mathcal{RBF}_{\omega}(B),\ \boldsymbol{X}^B(f,g)\leq\sqrt{10}\mathfrak{C}\text{ and }\left(\boldsymbol{X}^B\right)'(f,g)\leq\sqrt{10}\mathfrak{C}.\end{aligned}$$

Now each form $\mathsf{B}_{\Supset,B}^{A}(f_B,g_B)$ has its 'top' function $g_B=\frac{1}{\alpha_{\mathcal{B}}(B)}\mathsf{P}_{\mathcal{C}_B^{\omega}}^{\omega}g$ in $\mathcal{RBF}_{\omega}(B)$, and its 'bottom' function $f_B=\mathsf{P}_{\mathcal{C}_A^{\sigma}\cap\mathcal{C}_B^{\omega}}^{\sigma}f$ in $\mathcal{MBF}_{\sigma}(B)$, and finally the pair satisfies the stopping energy bound in (6.2). We say that such a form is of type $\mathcal{RBF}/\mathcal{MBF}$.

Thus we have reduced matters to bounding forms of type $\mathcal{MBF}/L^2$ and $\mathcal{RBF}/\mathcal{MBF}$. Note that in both inequalities, the lower function has only its L^2 norm appearing on the right hand side, without the measure of its supporting set.

6.1. **Reduction to stopping forms.** Now the boundedness of $\mathsf{B}_{\Subset,B}$ reduces to boundedness of the three terms B_1, B_2 and B_3 on page 11 of [LaSaShUr]. Here the term B_2 is controlled by the NTV constant $\mathfrak{NTV}$, term B_1 is controlled by the functional energy constant $\mathfrak{F}$, which by the Functional Energy Proposition 4.22 in this paper is controlled by $\mathfrak{NTV}$, and finally where the term B_3 is the form,

$$B_3(f,g) \equiv \sum_{I\in\mathcal{F}} \sum_{J:\ J\Subset I \text{ and } I_J=I_{big}} \left(\mathbb{E}_{I_J}^{\sigma} \triangle_I^{\sigma} f\right) \left\langle H_\sigma \mathbf{1}_{I_J}, \triangle_J^{\omega} g\right\rangle_{\omega},$$

where $\mathcal{F}$ is the Haar support of f. We are here considering the case $I_J = I_{big}$, because when $I_J = I_{small}$, we can simply use that the I_J are pairwise disjoint. Note that here f is, when appropriately normalized, a minimal bounded fluctuation function on A, while $g = \mathsf{P}_{\mathcal{C}_{\widetilde{A}}^{\omega}}^{\omega} g$ is in $\widetilde{L}_A^2(\omega)$, and because of the restriction $J \Subset I$ and $I_J = I_{big}$, the function g lies 'underneath' f.

In similar fashion, the boundedness of the form $\mathsf{B}_{\Supset,B}$ reduces to

$$B_3'(f_B,g_B) \equiv \sum_{J\in\mathcal{G}\cap\mathcal{C}_B^{\omega}} \sum_{I:\ I\Subset J \text{ and } J_I=J_{big}} \left(\mathbb{E}_{J_I}^{\omega} \triangle_J^{\omega} g_B\right) \left\langle \triangle_I^{\sigma} f_B^{\maltese}, H_\omega \mathbf{1}_{J_I}\right\rangle_{\sigma},$$

where $\mathcal{G}$ is the Haar support of g. Note that here $g_B = \mathsf{P}_{\mathcal{C}_B^{\omega}}^{\omega} g$ is, when appropriately normalized, a restricted bounded fluctuation function on B, while $f_B = \mathsf{P}_{\mathcal{C}_A^{\sigma}\cap\mathcal{C}_B^{\omega}}^{\sigma} f$ is, when appropriately normalized, a minimal bounded fluctuation function on B, and because of the restriction $I \Subset J$ and $J_I = J_{big}$, the function f_B lies 'underneath' g_B.

We now use the 'paraproduct' trick of NTV, namely that boundedness of $B_3(f,g)$ is equivalent to boundedness of the stopping form

$$\mathsf{B}_{stop}(f,g) \equiv \sum_{I\in\mathcal{F}} \sum_{J:\ J\Subset I \text{ and } I_J=I_{big}} \left(\mathbb{E}_{I_J}^{\sigma} \triangle_I^{\sigma} f\right) \left\langle H_\sigma \mathbf{1}_{S\setminus I_J}, \triangle_J^{\omega} g\right\rangle_{\omega},$$

and similarly, that boundedness of $B_3'(f_B,g_B)$ is equivalent to boundedness of the stopping form $\mathsf{B}_{stop}'(f_B,g_B)$. Thus we must bound the stopping form $\mathsf{B}_{stop}(f,g)$ in two cases, $\mathcal{MBF}/L^2$ and $\mathcal{RBF}/\mathcal{MBF}$.

We emphasize that the Functional Energy condition defined in [LaSaShUr] uses Calderón-Zygmund stopping intervals to separate pairs of intervals, and is consequently *identical* to the Functional Energy condition (2.6) defined here.

The above considerations have reduced the two weight inequality (1.1) for the Hilbert transform to the following two inequalities involving the highly nonlinear form B_{stop}.

Lemma 6.4. *The two weight Hilbert transform inequality (1.1) is implied by the following nonlinear inequalities with best constants $\mathfrak{B}_{stop}^{\mathcal{MBF}/L^2}$ and $\mathfrak{B}_{stop}^{\mathcal{RBF}/\mathcal{MBF}}$:*

$$\begin{aligned} &|\mathsf{B}_{stop}(f,g)| \lesssim \mathfrak{B}_{stop}^{\mathcal{MBF}/L^2}\left(\|f\|_{L^2(\sigma)} + \sqrt{|A|_\sigma}\right)\|g\|_{L^2(\omega)}, \qquad (6.5)\\ &\text{for } f \in \mathcal{MBF}_\sigma^{(\gamma)}(A),\ g \in \widetilde{L}_A^2(\omega) \text{ and } \mathbf{X}^A(f,g) \le \sqrt{10}\mathfrak{C}, \end{aligned}$$

and

$$(6.6)\quad \left|\mathsf{B}_{stop}(f,g)\right| \lesssim \mathfrak{B}_{stop}^{\mathcal{RBF}/\mathcal{MBF}}\left(\|f\|_{L^2(\sigma)}+\sqrt{|A|_\sigma}\right)\|g\|_{L^2(\omega)},$$

$$\text{for } f \in \mathcal{RBF}_\sigma^{(\gamma)}(A),\ g\in\mathcal{MBF}_\omega^{(\gamma)}(A) \text{ and } \mathbf{X}^A(f,g)\le\sqrt{10}\mathfrak{C},$$

along with their 'dual' formulations with best constants $\mathfrak{B}_{stop}^{\mathcal{MBF}/L^2\ *}$ *and* $\mathfrak{B}_{stop}^{\mathcal{RBF}/\mathcal{MBF}\ *}$.

Note again that as observed above, the lower function g has only its $L^2(\omega)$ norm appearing on the right hand side. The first inequality (6.5) is taken up in the next subsection. The second inequality then follows almost immediately, and is treated in the final subsection.

6.2. **Boundedness of the $\mathcal{MBF}/L^2$ stopping form.** We show that $\mathfrak{B}_{stop}^{\mathcal{MBF}/L^2}$ is controlled by the NTV constant $\mathfrak{NTV}$.

Proposition 6.7. *Let σ and ω be locally finite positive Borel measures on the real line $\mathbb{R}$ with no common point masses. Then*

$$\mathfrak{B}_{stop}^{\mathcal{MBF}/L^2}\lesssim\mathfrak{NTV}.$$

Proof: Let $\mathcal{F}$ and $\mathcal{G}$ denote the Haar supports of $f\in\mathcal{MBF}_\sigma^{(\gamma)}(A)$ and $g\in\widetilde{L}_A^2(\omega)$ respectively. Define

$$\mathcal{P}(f,g)\equiv\left\{(I,J)\in\mathcal{F}\times\mathcal{G}: J\Subset I \text{ and } I_J=I_{big}\right\}.$$

Then

$$\begin{aligned}\mathsf{B}_{stop}(f,g) &\equiv \sum_{I\in\mathcal{F}}\ \sum_{J:\ J\Subset I \text{ and } I_J=I_{big}}\left(\mathbb{E}_{I_J}^\sigma\triangle_I^\sigma f\right)\left\langle H_\sigma\mathbf{1}_{S\setminus I_J},\triangle_J^\omega g\right\rangle_\omega\\ &= \sum_{(I,J)\in\mathcal{P}(f,g)}\left(\mathbb{E}_{I_J}^\sigma\triangle_I^\sigma f\right)\left\langle H_\sigma\mathbf{1}_{S\setminus I_J},\triangle_J^\omega g\right\rangle_\omega.\end{aligned}$$

Given a subset $\mathcal{P}$ of $\mathcal{P}(f,g)$ we define

$$\mathsf{B}_{stop}^{\mathcal{P}}(f,g)\equiv\sum_{(I,J)\in\mathcal{P}}\left(\mathbb{E}_{I_J}^\sigma\triangle_I^\sigma f\right)\left\langle H_\sigma\mathbf{1}_{S\setminus I_J},\triangle_J^\omega g\right\rangle_\omega$$

and

$$\mathbf{size}(\mathcal{P})\equiv\sup_{I_1:\ I_1\subset A}\left(\sum_{\substack{I:\text{ there is }(I,J)\in\mathcal{P}\\ J\subset I_1\subset I}}\mathbb{E}_{I_J}^\sigma\triangle_I^\sigma f\right).$$

Clearly we have $\mathbf{size}(\mathcal{P})\le 1$ since $f\in\mathcal{MBF}_\sigma^{(\gamma)}(A)$.

Here is the main lemma.

Lemma 6.8. *Given $\mathcal{P}\subset\mathcal{P}(f,g)$, there are subsets $\mathcal{P}_{big}$ and $\mathcal{P}_{small}$ of $\mathcal{P}$ such that*

$$\begin{aligned}\mathcal{P} &= \mathcal{P}_{big}\dot\cup\mathcal{P}_{small},\\ \left|\mathsf{B}_{stop}^{\mathcal{P}_{big}}(f,g)\right| &\lesssim (\mathfrak{NTV})\ \mathbf{size}(\mathcal{P})\ \sqrt{|A|_\sigma}\ \|g\|_{L^2(\omega)},\\ \mathbf{size}(\mathcal{P}_{small}) &\le \frac{3}{4}\mathbf{size}(\mathcal{P}).\end{aligned}$$

This lemma proves Proposition 6.7 as follows. Apply Lemma 6.8 to $\mathcal{P}_0 \equiv \mathcal{P}(f,g)$ to obtain $(\mathcal{P}_0)_{big}$ and $(\mathcal{P}_0)_{small}$. Then apply Lemma 6.8 to $\mathcal{P}_1 \equiv (\mathcal{P}_0)_{small}$ to obtain $(\mathcal{P}_1)_{big}$ and $(\mathcal{P}_1)_{small}$. Continue by induction to define $\mathcal{P}_m \equiv (\mathcal{P}_{m-1})_{small}$ for $m \geq 1$. Then Lemma 6.8 gives

$$\begin{aligned} \mathbf{size}(\mathcal{P}_m) &= \mathbf{size}((\mathcal{P}_{m-1})_{small}) \leq \frac{3}{4}\mathbf{size}(\mathcal{P}_{m-1}) \\ &\leq ... \leq \left(\frac{3}{4}\right)^m \mathbf{size}(\mathcal{P}_0) \leq \left(\frac{3}{4}\right)^m , \end{aligned}$$

and so also

$$\begin{aligned} \left|\mathsf{B}_{stop}^{(\mathcal{P}_m)_{big}}(f,g)\right| &\lesssim (\mathfrak{NTV})\ \mathbf{size}(\mathcal{P}_m)\ \sqrt{|A|_\sigma}\ \|g\|_{L^2(\omega)} \\ &\lesssim (\mathfrak{NTV})\ \left(\frac{3}{4}\right)^m \sqrt{|A|_\sigma}\ \|g\|_{L^2(\omega)} . \end{aligned}$$

Since $\mathcal{P}(f,g) = \bigcup_{m=1}^{\infty} (\mathcal{P}_m)_{big}$, we thus have

$$\begin{aligned} |\mathsf{B}_{stop}(f,g)| &= \left|\sum_{m=1}^{\infty} \mathsf{B}_{stop}^{(\mathcal{P}_m)_{big}}(f,g)\right| \\ &\lesssim \sum_{m=1}^{\infty} (\mathfrak{NTV})\ \left(\frac{3}{4}\right)^m \sqrt{|A|_\sigma}\ \|g\|_{L^2(\omega)} \\ &\lesssim (\mathfrak{NTV})\ \sqrt{|A|_\sigma}\ \|g\|_{L^2(\omega)} . \end{aligned}$$

Proof. (of Lemma 6.8) The two key properties of $f \in \mathcal{MBF}_\sigma^{(\gamma)}(A)$ that we will use are

$$\mathbb{E}_{I_J}^\sigma \triangle_I^\sigma f \geq 0 \text{ and } \sum_{I:\ J\subset I\subset A} \mathbb{E}_{I_J}^\sigma \triangle_I^\sigma f \leq 1.$$

Consider those intervals I_1 that are maximal subject to the condition,

$$\sum_{\substack{I:\text{ there is }(I,J)\in\mathcal{P} \\ J\subset I_1\subset I}} \mathbb{E}_{I_J}^\sigma \triangle_I^\sigma f \geq \frac{3}{4}\mathbf{size}(\mathcal{P}),$$

and choose one of them with maximum length. Define

$$\begin{aligned} \mathcal{R}_1 &\equiv \{(I,J) \in \mathcal{P} : J \subset I_1 \subset I\}; \\ \mathcal{P}_1 &\equiv \mathcal{P} \setminus \mathcal{R}_1. \end{aligned}$$

If $I_1, ..., I_{m-1}$ have been chosen, then consider those intervals I_m that are maximal subject to the condition,

$$\sum_{\substack{I:\text{ there is }(I,J)\in\mathcal{P}_{m-1} \\ J\subset I_m\subset I}} \mathbb{E}_{I_J}^\sigma \triangle_I^\sigma f \geq \frac{3}{4}\mathbf{size}(\mathcal{P}_{m-1}),$$

and choose one of them with maximum length. Define

$$\begin{aligned} \mathcal{R}_m &\equiv \{(I,J) \in \mathcal{P}_{m-1} : J \subset I_m \subset I\}; \\ \mathcal{P}_m &\equiv \mathcal{P}_{m-1} \backslash \mathcal{R}_m. \end{aligned}$$

It is easy to see that the collection of intervals $\{I_m\}_{m=1}^{\infty}$ is pairwise disjoint. Indeed, this follows from the choice of parameter $\frac{3}{4}$ in the maximal conditions.

Next, we define

$$\mathcal{P}_{small} \equiv \mathcal{P} \setminus \left(\bigcup_{m=1}^{\infty} \mathcal{R}_m \right) ,$$

and we have the inequality

$$\mathbf{size}\left(\mathcal{P}_{small}\right) \leq \frac{3}{4}\mathbf{size}\left(\mathcal{P}\right).$$

Now we have

$$\begin{aligned}
\mathsf{B}_{stop}^{\mathcal{R}_m}(f,g) &= \sum_{(I,J)\in\mathcal{R}_m} \left(\mathbb{E}_{I_J}^{\sigma} \triangle_I^{\sigma} f\right) \left\langle H_\sigma \mathbf{1}_{S\setminus I_J}, \triangle_J^{\omega} g\right\rangle_\omega \\
&\leq \left(\sum_{I:\ I_m\subset I\subset A} \mathbb{E}_{I_J}^{\sigma} \triangle_I^{\sigma} f\right) \left\langle H_\sigma \mathbf{1}_{S\setminus I_J}, \sum_{J:\ J\subset I_m} \triangle_J^{\omega} g\right\rangle_\omega \\
&\leq \left\langle H_\sigma \mathbf{1}_{S\setminus I_J}, \sum_{J:\ J\subset I_m} \triangle_J^{\omega} g\right\rangle_\omega
\end{aligned}$$

since $\mathbb{E}_{I_J}^{\sigma} \triangle_I^{\sigma} f \geq 0$ and $\sum_{I:\ I_m\subset I\subset A} \mathbb{E}_{I_J}^{\sigma} \triangle_I^{\sigma} f \leq 1$. From the monotonicity property and the energy bound, it now follows that

$$\left|\mathsf{B}_{stop}^{\mathcal{R}_m}(f,g)\right| \lesssim (\mathfrak{NTV}) \sqrt{|I_m|_\sigma}\ \|g_{I_m}\|_{L^2(\omega)} ,$$

where $g_{I_m} \equiv \sum_{J:\ J\subset I_m} \triangle_J^{\omega} g$.

Thus with

$$\mathcal{P}_{big} \equiv \bigcup_{m=1}^{\infty} \mathcal{R}_m ,$$

we get

$$\begin{aligned}
\left|\mathsf{B}_{stop}^{\mathcal{P}_{big}}(f,g)\right| &\lesssim \sum_{m=1}^{\infty} \left|\mathsf{B}_{stop}^{\mathcal{R}_m}(f,g)\right| \\
&\lesssim \sum_{m=1}^{\infty} (\mathfrak{NTV}) \sqrt{|I_m|_\sigma}\ \|g_{I_m}\|_{L^2(\omega)} \\
&\lesssim (\mathfrak{NTV}) \left(\sum_{m=1}^{\infty} |I_m|_\sigma\right)^{\frac{1}{2}} \left(\sum_{m=1}^{\infty} \|g_{I_m}\|_{L^2(\omega)}^2\right)^{\frac{1}{2}} \\
&\lesssim (\mathfrak{NTV}) \sqrt{|A|_\sigma}\, \|g\|_{L^2(\omega)} ,
\end{aligned}$$

since the intervals I_m are pairwise disjoint, and the functions $g_{I_m} \equiv \triangle_J^{\omega} g$ are thus mutually orthogonal. □

6.3. **Boundedness of the $\mathcal{RBF}/\mathcal{MBF}$ stopping form .** We have already done all the work needed to bound the $\mathcal{RBF}/\mathcal{MBF}$ stopping form. Indeed, we have proved above the following inequalities for $f \in \mathcal{RBF}_\sigma^{(\gamma)}(A)$ and $g \in \mathcal{MBF}_\omega^{(\gamma)}(A)$, when we have both the energy stopping bound $\boldsymbol{X}^A(f,g) \leq \sqrt{10}\mathfrak{C}$ and the dual energy stopping bound $\left(\boldsymbol{X}^A\right)'(f,g) \leq \sqrt{10}\mathfrak{C}$:

$$\begin{aligned}
|H_\sigma(f,g)| &\lesssim \mathfrak{NTV}_{ind} \|f\|_{L^2(\sigma)} \|g\|_{L^2(\omega)};\\
|H_\sigma(f,g) - \{\mathsf{B}_{\Subset,A}(f,g) + \mathsf{B}_{\Supset,A}(f,g)\}| &\lesssim \mathfrak{NTV} \|f\|_{L^2(\sigma)} \|g\|_{L^2(\omega)};\\
\left|\mathsf{B}^A_{\Subset,stop}(f,g) - \mathsf{B}_{\Subset,A}(f,g)\right| &\lesssim \mathfrak{NTV} \|f\|_{L^2(\sigma)} \|g\|_{L^2(\omega)};\\
\left|\mathsf{B}^A_{\Supset,stop}(f,g) - \mathsf{B}_{\Supset,A}(f,g)\right| &\lesssim \mathfrak{NTV} \|f\|_{L^2(\sigma)} \|g\|_{L^2(\omega)};\\
\left|\mathsf{B}^A_{\Supset,stop}(f,g)\right| &\lesssim \mathfrak{NTV} \|f\|_{L^2(\sigma)} \|g\|_{L^2(\omega)};
\end{aligned}$$

where we are writing $\mathsf{B}^A_{\Subset,stop}$ for the stopping form corresponding to $\mathsf{B}_{\Subset,A}$, and $\mathsf{B}^A_{\Supset,stop}$ for the stopping form corresponding to $\mathsf{B}_{\Supset,A}$. The crucial final inequality follows from Proposition 6.7, because in the form $\mathsf{B}^A_{\Supset,stop}(f,g)$, it is $g \in \mathcal{MBF}^{(\gamma)}_\omega(A)$ that is the function on top. Thus we conclude that for $f \in \mathcal{RBF}^{(\gamma)}_\sigma(A)$, $g \in \mathcal{MBF}^{(\gamma)}_\omega(A)$, $\mathbf{X}^A(f,g) \leq \sqrt{10}\mathfrak{C}$ and $\left(\mathbf{X}^A\right)'(f,g) \leq \sqrt{10}\mathfrak{C}$, we have

$$\begin{aligned}
\mathsf{B}^A_{\Subset,stop}(f,g) = \; & \mathsf{B}^A_{\Subset,stop}(f,g) - \mathsf{B}_{\Subset,A}(f,g)\\
&+\mathsf{B}_{\Subset,A}(f,g) + \mathsf{B}_{\Supset,A}(f,g) - H_\sigma(f,g)\\
&+H_\sigma(f,g)\\
&-\mathsf{B}_{\Supset,A}(f,g) + \mathsf{B}^A_{\Supset,stop}(f,g)\\
&-\mathsf{B}^A_{\Supset,stop}(f,g),
\end{aligned}$$

and so $\left|\mathsf{B}^A_{\Subset,stop}(f,g)\right|$ is bounded by $C\mathfrak{NTV}_{ind} \|f\|_{L^2(\sigma)} \|g\|_{L^2(\omega)}$.

7. Appendix

7.1. **Equivalence of indicator/interval testing conditions.** It is easy to see that the bounded function f in (1.3) can be replaced by χ_E for every compact subset E of I. Indeed, given $g \in L^2(\omega)$ and I an interval, define $F_{g,I} = \frac{H_\omega(\chi_I g)}{|H_\omega(\chi_I g)|}$. Then we have

$$\begin{aligned}
&\sup_{|f|\leq 1}\left(\int_I H_\sigma(\chi_I f)^2 \omega\right)^{\frac{1}{2}} = \sup_{|f|\leq 1}\sup_{\|g\|_{L^2(\omega)}\leq 1}\left|\int_I H_\sigma(\chi_I f) g\omega\right|\\
=\; & \sup_{\|g\|_{L^2(\omega)}\leq 1}\sup_{|f|\leq 1}\left|\int_I H_\omega(\chi_I g) f\sigma\right| = \sup_{\|g\|_{L^2(\omega)}\leq 1}\int_I H_\omega(\chi_I g) F_{g,I}\sigma\\
=\; & \sup_{\|g\|_{L^2(\omega)}\leq 1}\int_I H_\sigma(\chi_I F_{g,I}) g\omega \leq \sup_{\|g\|_{L^2(\omega)}\leq 1}\left(\int_I H_\sigma(\chi_I F_{g,I})^2 \omega\right)^{\frac{1}{2}}.
\end{aligned}$$

Since $F_{g,I}$ takes on only the values ± 1, it is easy to see that we can take $f = \chi_E$ in (1.3) if we double the constant.

7.2. **Infinite bounded fluctuation decomposition.** Here we show that the number of terms in the decomposition in Proposition 5.8 cannot be bounded, thus necessitating the elaborate decomposition there. For $n \geq 0$, define dyadic subintervals I_n, J_n of $[0,1]$ by

$$\begin{aligned} I_n &\equiv \left[0, 2^{-n}\right]; \\ J_n &\equiv I_n \setminus I_{n+1} = \left[2^{-n-1}, 2^{-n}\right], \end{aligned}$$

and define a positive measure σ on $[0,1]$ by

$$\sigma = \sum_{k=0}^{\infty} \left(\sigma_k \frac{1}{|J_{2k}|} \mathbf{1}_{J_{2k}} + \tau_k \frac{1}{|J_{2k+1}|} \mathbf{1}_{J_{2k+1}} \right).$$

Then define positive functions f, g and $h = f + g$ on $[0,1]$ by

$$f = \sum_{k=0}^{\infty} f_k \mathbf{1}_{J_{2k}}, \quad g = \sum_{k=0}^{\infty} g_k \mathbf{1}_{J_{2k+1}}.$$

Lemma 7.1. *If*

$$f_k = \gamma > 2, \qquad g_k = \frac{1}{2}, \qquad \sigma_k = \frac{1}{2\gamma} \tau_k,$$

for all k, then

$$\widetilde{h} \equiv \frac{1}{2} \left\{ h - \left(\mathbb{E}^{\sigma}_{[0,1]} h \right) \mathbf{1}_{[0,1]} \right\} \in \mathcal{BF}^{\left(\frac{\gamma}{2}\right)}_{\sigma} ([0,1]).$$

Proof. We compute using

$$\gamma \sigma_k + \frac{1}{2} \tau_k = \tau_k = \frac{\sigma_k + \tau_k}{\frac{1}{2\gamma} + 1},$$

that

$$\begin{aligned} \int_{I_{2\ell}} h\sigma &= \sum_{k=\ell}^{\infty} (f_k \sigma_k + g_k \tau_k) = \sum_{k=\ell}^{\infty} \left(\gamma \sigma_k + \frac{1}{2} \tau_k \right) \\ &= \sum_{k=\ell}^{\infty} \frac{\sigma_k + \tau_k}{\frac{1}{2\gamma} + 1} = \frac{1}{\frac{1}{2\gamma} + 1} \int_{I_{2\ell}} \sigma. \end{aligned}$$

We also have

$$\begin{aligned} \int_{I_{2\ell+1}} h\sigma &= \sum_{k=\ell+1}^{\infty} f_k \sigma_k + \sum_{k=\ell}^{\infty} g_k \tau_k = \frac{1}{2} \tau_\ell + \sum_{k=\ell+1}^{\infty} \left(\gamma \sigma_k + \frac{1}{2} \tau_k \right) \\ &= \frac{1}{2} \tau_\ell + \sum_{k=\ell+1}^{\infty} \frac{\sigma_k + \tau_k}{\frac{1}{2\gamma} + 1} = \frac{1}{2} \tau_\ell + \frac{1}{\frac{1}{2\gamma} + 1} \int_{I_{2\ell+2}} \sigma \\ &= \frac{1}{2} \int_{J_{2\ell+1}} \sigma + \frac{1}{\frac{1}{2\gamma} + 1} \int_{I_{2\ell+2}} \sigma. \end{aligned}$$

Thus

$$\mathbb{E}^{\sigma}_{I_{2\ell}} h = \frac{1}{\frac{1}{2\gamma} + 1}, \qquad \mathbb{E}^{\sigma}_{I_{2\ell+1}} h = \frac{\frac{1}{2} \int_{J_{2\ell+1}} \sigma + \frac{1}{\frac{1}{2\gamma}+1} \int_{I_{2\ell+2}} \sigma}{\int_{J_{2\ell+1}} \sigma + \int_{I_{2\ell+2}} \sigma}.$$

In particular, $\mathbb{E}^{\sigma}_{I} h \leq 1$ on I_n, $n \geq 0$, and since h is constant on every other dyadic subinterval of $[0,1]$, we conclude that $\widetilde{h} \in \mathcal{BF}^{\left(\frac{\gamma}{2}\right)}_{\sigma} ([0,1])$. $\square$

Lemma 7.2. *Let f, g and h be as above, and let $\frac{\gamma-1}{2} > 2C > C > 1$ be fixed positive constants. If there is a decomposition*

$$\widetilde{h} = F + G$$

where $F \in \mathcal{RBF}_\sigma^{\left(\frac{\gamma-1}{4}\right)}([0,1])$ and G is bounded by C, then

$$F = \sum_{\ell=0}^{\infty} \triangle_{I_{2\ell}}^{\sigma} \widetilde{h}.$$

Proof. The Haar support of $h - \mathbb{E}_{[0,1]}^{\sigma} h$ is contained in the collection of intervals

$$\{I_n\}_{n=0}^{\infty} = \{I_{2\ell}\}_{\ell=0}^{\infty} \dot{\cup} \{I_{2\ell+1}\}_{\ell=0}^{\infty} \equiv \mathcal{K}_{even} \dot{\cup} \mathcal{K}_{odd}.$$

Since G is bounded by C on $I_{2\ell}$, and $h = \gamma$ on $J_{2\ell}$, we must have $F = \widetilde{h} - G > \frac{\gamma}{2}$ on $J_{2\ell}$, and it follows that the Haar support of F contains $\mathcal{K}_{even}$. Now on the intervals $I_{2\ell+1}$, the average of $\left|\widetilde{h}\right|$ on each child of $I_{2\ell+1}$ is at most 1, and it follows that $\mathcal{K}_{odd}$ is disjoint from the Haar support of F (we need γ sufficiently large here, e.g. $\gamma > 4C + 5$). Thus we have that the Haar support of F is precisely $\mathcal{K}_{even}$. $\square$

Thus if f, g and h are as above, we conclude that

$$G = \sum_{\ell=0}^{\infty} \triangle_{I_{2\ell+1}}^{\sigma} \widetilde{h}.$$

We now compute that on the left child $I_{2\ell+2}$ of $I_{2\ell+1}$ we have

$$\begin{aligned}
\triangle_{I_{2\ell+1}}^{\sigma} \widetilde{h} &= \mathbb{E}_{I_{2\ell+2}}^{\sigma} h - \mathbb{E}_{I_{2\ell+1}}^{\sigma} h = \frac{1}{\frac{1}{2\gamma}+1} - \frac{\frac{1}{2}\int_{J_{2\ell+1}} \sigma + \frac{1}{\frac{1}{2\gamma}+1}\int_{I_{2\ell+2}} \sigma}{\int_{J_{2\ell+1}} \sigma + \int_{I_{2\ell+2}} \sigma} \\
&= \frac{\left(\int_{J_{2\ell+1}} \sigma + \int_{I_{2\ell+2}} \sigma\right) - \left(\frac{1}{2\gamma}+1\right)\left(\frac{1}{2}\int_{J_{2\ell+1}} \sigma + \frac{1}{\frac{1}{2\gamma}+1}\int_{I_{2\ell+2}} \sigma\right)}{\left(\frac{1}{2\gamma}+1\right)\int_{I_{2\ell+1}} \sigma} \\
&= \frac{\left(\frac{1}{2} - \frac{1}{4\gamma}\right)\int_{J_{2\ell+1}} \sigma}{\left(\frac{1}{2\gamma}+1\right)\int_{I_{2\ell+1}} \sigma} = \frac{1}{2}\frac{1-\frac{1}{2\gamma}}{1+\frac{1}{2\gamma}} \frac{\int_{J_{2\ell+1}} \sigma}{\int_{I_{2\ell+1}} \sigma} = \frac{1}{2}\frac{1-\frac{1}{2\gamma}}{1+\frac{1}{2\gamma}} \frac{\tau_\ell}{\sum_{k=\ell}^{\infty} \tau_k + \sum_{k=\ell+1}^{\infty} \sigma_k},
\end{aligned}$$

so that

$$\begin{aligned}
2\frac{1+\frac{1}{2\gamma}}{1-\frac{1}{2\gamma}} \mathbb{E}_{I_{2\ell+2}}^{\sigma}\left(\triangle_{I_{2\ell+1}}^{\sigma} \widetilde{h}\right) &= \frac{\tau_\ell}{\sum_{k=\ell}^{\infty} \tau_k + \sum_{k=\ell+1}^{\infty} \frac{1}{2\gamma}\tau_k} \\
&> \frac{\tau_\ell}{\sum_{k=\ell}^{\infty} \tau_k + \sum_{k=\ell}^{\infty} \frac{1}{2\gamma}\tau_k} = \frac{1}{1+\frac{1}{2\gamma}} \frac{\tau_\ell}{\sum_{k=\ell}^{\infty} \tau_k}.
\end{aligned}$$

Thus the left children have expected value that is positive and

$$\sum_{\ell=0}^{\infty} \left|\mathbb{E}_{I_{2\ell+2}}^{\sigma}\left(\triangle_{I_{2\ell+1}}^{\sigma} \widetilde{h}\right)\right| > \frac{1}{2}\frac{1-\frac{1}{2\gamma}}{\left(1+\frac{1}{2\gamma}\right)^2} \sum_{\ell=0}^{\infty} \frac{\tau_\ell}{\sum_{k=\ell}^{\infty} \tau_k} = \infty.$$

Note that the sum $\sum_{\ell=0}^{\infty} \frac{\tau_\ell}{\sum_{k=\ell}^{\infty} \tau_k}$ diverges for any decreasing sequence $\{\tau_k\}_{k=0}^{\infty}$ of positive numbers with finite sum (apply the integral test with $f(\ell) = \tau_\ell$ and use

$\int_0^\infty \frac{f(x)}{\int_x^\infty f} dx = \infty$). Thus we see that for $x \in J_{2n+1}$ we have

$$\begin{aligned} G(x) &= \sum_{\ell=0}^{\infty} \triangle_{I_{2\ell+1}}^{\sigma} \widetilde{h}(x) = \triangle_{I_{2n+1}}^{\sigma} \widetilde{h}(x) + \sum_{\ell=0}^{n-1} \triangle_{I_{2\ell+1}}^{\sigma} \widetilde{h}(x) \\ &= \mathbb{E}_{J_{2n+1}}^{\sigma} \widetilde{h} - \mathbb{E}_{I_{2n+1}}^{\sigma} \widetilde{h} + \sum_{\ell=0}^{n-1} \mathbb{E}_{I_{2\ell+2}}^{\sigma} \left(\triangle_{I_{2\ell+1}}^{\sigma} \widetilde{h} \right) > \frac{1-\frac{1}{2\gamma}}{\left(1+\frac{1}{2\gamma}\right)^2} \sum_{\ell=0}^{n-1} \frac{\tau_\ell}{\sum_{k=\ell}^{\infty} \tau_k}, \end{aligned}$$

which tends to ∞ as $x \to 0$.

Conclusion 7.3. *Suppose $\gamma > 4C+5, C > 1$. Then there exists $h \in \mathcal{BF}_\sigma^{(\gamma)}([0,1])$ such that there is no decomposition $h = F + G$ with $F \in \mathcal{RBF}_\sigma^{\left(\frac{\gamma-1}{4}\right)}([0,1])$ and $\|G\|_\infty \leq C$.*

Remark 7.4. *The measure σ can be taken comparable to Lebesgue measure, namely with $\tau_k = |J_{2k+1}| = 2^{-2k-2}$ and $\sigma_k = \frac{1}{2\gamma} 2^{-2k-2}$,*

$$\sigma = \sum_{k=0}^{\infty} \left(\sigma_k \frac{1}{|J_{2k}|} \mathbf{1}_{J_{2k}} + \tau_k \frac{1}{|J_{2k+1}|} \mathbf{1}_{J_{2k+1}} \right) = \sum_{k=0}^{\infty} \left(\frac{1}{4\gamma} \mathbf{1}_{J_{2k}} + \mathbf{1}_{J_{2k+1}} \right).$$

Note however that the doubling eccentricity of σ is 2γ, essentially the smallest eccentricity possible for the existence of a nontrivial function in $\mathcal{RBF}_\sigma^{\left(\frac{\gamma}{2}\right)}([0,1])$.

Remark 7.5. *The conclusion is easily extended to show that for each positive integer $N \in \mathbb{N}$, there is $h_N \in \mathcal{BF}_\sigma^{(\gamma)}([0,1])$ such that there is no decomposition $h_N = \sum_{n=1}^N F_n + \sum_{n=1}^N G_n$ with $F_n \in \mathcal{RBF}_\sigma^{\left(\frac{\gamma-1}{4}\right)}([0,1])$ and $\|G_n\|_\infty \leq C$ for each $1 \leq n \leq N$.*

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