A PRIORI ESTIMATES FOR QUASILINEAR EQUATIONS RELATED TO THE MONGE-AMPÈRE EQUATION IN TWO DIMENSIONS

ERIC T. SAWYER AND RICHARD L. WHEEDEN

ABSTRACT. We prove *a priori* inequalities for non-subelliptic quasilinear equations related to the Monge-Ampère equation in two dimensions, for example equations of the type

$$\mathcal{L}w = \partial_x^2 w + \partial_y \left[k\left(x, w\left(x,y\right)\right)\partial_y w\right] = 0.$$

1. INTRODUCTION

This paper is a companion to our paper [12]. The *a priori* estimates for quasilinear equations proved here are used in [12] to obtain regularity results for certain non-subelliptic generalized Monge-Ampère equations in two dimensions. More precisely, we give two types of *a priori* estimate here. The first type of estimate in Subsection 1.1 applies in a rather general setting, where ellipticity may degenerate to infinite order, and concludes that higher order derivatives of solutions can be controlled by the *first and zero order* derivatives. The second type of estimate in Subsection 1.2 applies to more restrictive equations, where the infinite degeneracy of ellipticity is balanced by a compensating linearity, and concludes that higher order derivatives can be controlled by the *zero order* derivatives alone. It is this latter estimate that finds application in [12]. Extensions of these estimates to higher dimensions will appear in a paper [10] in preparation with C. Rios. See also an earlier preprint [13] with additional detail.

1.1. *A priori* estimates in terms of ∇w.

Here we consider the degenerate quasilinear equation

$$\mathcal{L}w = \left[\partial_x^2 + \partial_y k\left(x, w\left(x,y\right)\right)\partial_y\right] w = 0, \qquad (x,y) \in \Omega', \tag{1.1}$$

where $k\left(x,y\right)$ is smooth (infinitely differentiable) and nonnegative in a domain Ω, and where $w\left(x,y\right)$ and Ω' are such that

$$\left(x, w\left(x,y\right)\right) \in \Omega \textit{ for all } (x,y) \in \Omega', \tag{1.2}$$

and where k is positive for $x \neq 0$. This is motivated by the Dirichlet problem for the Monge-Ampère equation,

$$\left\{ \begin{array}{rcll} u_{xx}u_{yy} - \left(u_{xy}\right)^2 & = & k\left(x,y\right), & (x,y) \in \Omega \\ u & = & \phi\left(x,y\right), & (x,y) \in \partial\Omega \end{array} \right. , \tag{1.3}$$

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where k and ϕ are smooth and Ω is a bounded convex planar domain with smooth positively curved boundary $\partial\Omega$. Indeed, as is shown in e.g. [12] or [14], the partial Legendre transformation $(s,t) = T(x,y)$ given by

$$\left\{ \begin{array}{lcl} s & = & x \\ t & = & u_y(x,y) \end{array} \right. , \tag{1.4}$$

where u is a convex $C^{1,1}$ solution of (1.3), reduces the question of interior regularity of solutions to (1.3) to the regularity of solutions of (1.1). Indeed, the function $y = y(s,t)$, given by inverting T when $k > 0$, is then a weak solution of

$$\left\{ \partial_s^2 + \partial_t k(s, y(s,t)) \partial_t \right\} y = 0.$$

Our first main theorem shows that if w is a smooth solution of (1.1) in Ω', then all its derivatives are controlled on compact subsets of Ω' by the size of w and ∇w (of course if Ω is bounded, then w is *a priori* bounded by the requirement (1.2)). Before stating this, it will be convenient to recall the classical inequality

$$|\nabla k(x,y)| \leq B\sqrt{k(x,y)}, \qquad (x,y) \in L, \tag{1.5}$$

for a compact subset L of Ω, and its more general form,

$$|\nabla k(x,y)| \leq C\left\{ \left\|\nabla^2 k\right\|_\infty^{\frac{1}{2}} + (dist((x,y),\partial\Omega))^{-\frac{1}{2}} \right\} \sqrt{k(x,y)}, \qquad (x,y) \in \Omega, \tag{1.6}$$

if k is merely nonnegative with bounded first and second derivatives on a domain Ω (see e.g. Appendix B in [12]). We will also need some notation. Let $\mathcal{P}_c(\Omega)$ denote the collection of all compact subsets of Ω. We will say that a positive function f defined on $\mathcal{P}_c(\Omega)$ is increasing if $f(L_1) \leq f(L_2)$ whenever $L_1, L_2 \in \mathcal{P}_c(\Omega)$ with $L_1 \subset L_2$.

Theorem 1.1. *Suppose $k(x,y)$ is smooth and nonnegative in a domain Ω, and is positive for $x \neq 0$. Let ζ and $\varkappa$ be smooth cutoff functions supported in Ω' with $\varkappa = 1$ on the support of ζ. Then for every multi-index α, there is a positive function $\mathcal{C}_\alpha(\sigma, L)$, defined for $(\sigma, L) \in [0,\infty) \times \mathcal{P}_c(\Omega)$ and increasing in each variable separately, depending only on Ω, Ω', $\sum_{|\beta| \leq |\alpha|+2} \left(\left\|D^\beta \zeta\right\|_\infty + \left\|D^\beta \varkappa\right\|_\infty \right)$, $\inf_{\{(x,y)\in L: |x| \geq \varepsilon_\alpha\}} k$ and $\sum_{|\beta| \leq |\alpha|+2} \left\|D^\beta k\right\|_{L^\infty(L)}$ where*

$$\varepsilon_\alpha = \varepsilon\left(\Omega, \|k\|_{C^{|\alpha|+2}(L)}, \left\| \frac{|\nabla k|}{k^{\frac{1}{2}}} \right\|_{L^\infty(L)} \right) > 0,$$

such that

$$\|\zeta D^\alpha w\|_\infty \leq \mathcal{C}_\alpha\left(\|\varkappa \nabla w\|_\infty, L \right) \tag{1.7}$$

for all smooth solutions w of (1.1) in Ω' such that $(x, w(x,y)) \in L$ for all (x,y) in the support of $\varkappa$.

Note that the right hand side of (1.7) depends implicitly on $\|\varkappa w\|_\infty$ through the restriction that $(x, w(x,y)) \in L$ when $\varkappa(x,y) \neq 0$.

Remark 1.1. *The important point in the above* a priori *estimate is that the dependence of the final bound in (1.7) on the function k involves only the size of derivatives of k on L, the rate of decay of k on L as $x \to 0$, and the constant B in (1.5), which also depends on L. In particular, these bounds are uniform over the family of functions $\{k+\delta\}_{0<\delta\leq 1}$ for k satisfying the hypotheses of the theorem. This observation provides the means of showing that the standard approximation* *procedure for the Monge-Ampère equation converges appropriately. See our companion paper* [12] *for details.*

More generally we consider the quasilinear system for three unknown functions w, r and z of two variables (x,y) in a plane set Ω';

$$\left\{\begin{array}{lcl}\partial_x^2 w+\partial_y k\left(x,w,r,z,y\right)\partial_y w & = & 0\\ \partial_x r-z-y\partial_x w & = & 0\\ \partial_y r-y\partial_y w & = & 0\\ \partial_x z-k\partial_y w & = & 0\\ \partial_y z+\partial_x w & = & 0\end{array}\right., \tag{1.8}$$

where $k\left(x,y,v,p,q\right)$ is smooth and nonnegative in $\Omega\times\mathbb{R}^3$ $(\Omega\subset\mathbb{R}^2)$, and

$$\left(x,w\left(x,y\right),r\left(x,y\right),z\left(x,y\right),y\right)\in\Omega\times\mathbb{R}^3 \text{ for } \left(x,y\right)\in\Omega'.$$

This system is motivated by the Dirichlet problem for the generalized Monge-Ampère equation,

$$\left\{\begin{array}{rcll}u_{xx}u_{yy}-\left(u_{xy}\right)^2 & = & k\left(x,y,u,u_x,u_y\right), & \left(x,y\right)\in\Omega\\ u & = & \phi\left(x,y\right), & \left(x,y\right)\in\partial\Omega\end{array}\right. \tag{1.9}$$

where $k\left(x,y,v,p,q\right)$ is smooth and nonnegative on $\overline{\Omega}\times\mathbb{R}^3$. As before, if $k>0$, we apply the partial Legendre transform associated to a smooth solution u of (1.9). As shown in [12], the functions

$$\left\{\begin{array}{lclcl}w & = & y & = & y\left(s,t\right)\\ z & = & u_x\left(x,y\right) & = & u_x\left(s,y\left(s,t\right)\right)\\ r & = & u\left(x,y\right) & = & u\left(s,y\left(s,t\right)\right)\end{array}\right.,$$

where $\left(x,y\right)=\left(s,y\left(s,t\right)\right)$ is the inverse partial Legendre transform, then satisfy the system (1.8) in the weak sense (where we have rewritten the independent variables s and t as x and y). The first order equations in (1.8) show that the (x,y) derivatives of z and r satisfy the same or better size estimates as do those of w, provided the sup norms of w, z and r are all *a priori* bounded (of course, only the bound on z is needed for this purpose). This is indeed the case for solutions arising from the partial Legendre transform by the *a priori* estimates for first order derivatives in [1] (which require only that $k\geq 0$). As a result, we have the following generalization of Theorem 1.1 with essentially the same proof.

Theorem 1.2. *Suppose $k\left(x,y,v,p,q\right)$ is smooth and nonnegative in a domain $\Omega\times\mathbb{R}^3$ and is positive for $x\neq 0$. Let ζ and $\varkappa$ be smooth cutoff functions supported in Ω' with $\varkappa=1$ on the support of ζ. Then for every multi-index α, there is a real-valued function $\mathcal{C}_\alpha\left(\sigma,L\right)$, defined for $\left(\sigma,L\right)\in[0,\infty)\times\mathcal{P}_c\left(\Omega\times\mathbb{R}^3\right)$ and increasing in each variable separately, depending only on Ω, Ω', $\sum_{|\beta|\leq|\alpha|+2}\left(\left\|D^\beta\zeta\right\|_\infty+\left\|D^\beta\varkappa\right\|_\infty\right)$, $\inf_{\{(x,y,v,p,q)\in L:|x|\geq\varepsilon_\alpha\}}k$ and $\sum_{|\beta|\leq|\alpha|+2}\left\|D^\beta k\right\|_{L^\infty(L)}$ where*

$$\varepsilon_\alpha=\varepsilon\left(\Omega,\|k\|_{C^{|\alpha|+2}(L)},\left\|\frac{|\nabla k|}{k^{\frac{1}{2}}}\right\|_{L^\infty(L)}\right)>0,$$

such that

$$\left\|\zeta D^\alpha w\right\|_\infty\leq\mathcal{C}_\alpha\left(\left\|\varkappa\nabla w\right\|_\infty,L\right)$$

for all smooth solutions w, z and r of (1.8) in Ω' such that

$$\left(x,w\left(x,y\right),r\left(x,y\right),z\left(x,y\right),y\right)\in L$$

for all (x,y) in the support of $\varkappa$.

1.2. ***A priori* estimates in terms of w.** We now consider the question of when we can improve the bound $\|\zeta D^{\alpha}w\|_{\infty}\leq\mathcal{C}_{\alpha}\left(\|\varkappa\nabla w\|_{\infty},L\right)$ in Theorem 1.2 to a bound $\|\zeta D^{\alpha}w\|_{\infty}\leq\mathcal{C}_{\alpha}\left(L\right)$ that does *not* depend on the size $\|\varkappa\nabla w\|_{\infty}$ of the gradient of w, but only on the size of $\|\varkappa w\|_{\infty}$ through the restriction that $(x,w)\in L$ for all (x,y) in the support of $\varkappa$. Such an improvement is necessary for the application to the generalized Monge-Ampère equation (1.9) in [12]. To achieve this we impose additional conditions on $k\left(x,y,v,p,q\right)$ which force it to become less dependent on the variables y, v, p and q as k goes to zero, namely

$$\begin{aligned}|k_i| &\leq Ck^{d(i)}, \quad 2\leq i\leq 4,\\ |k_{55}| &\leq Ck^{\frac{1}{2}},\end{aligned}\tag{1.10}$$

on compact subsets of $\Omega\times\mathbb{R}^3$, where k_j denotes differentiation with respect to the j^{th} of the 5 variables x,y,v,p,q, and[1]

$$d\left(i\right)=\begin{cases}\frac{3}{2}, & i=2,3\\ 1, & i=4\end{cases}.$$

Remark 1.2. *In the classical case where $k=k\left(x,y\right)$, (1.10) reduces to the single condition*

$$|k_2|\leq Ck^{\frac{3}{2}}.\tag{1.11}$$

In the equation for prescribing Gaussian curvature $\mathcal{K}\left(x,y\right)$, we have

$$k\left(x,y,v,p,q\right)=\mathcal{K}\left(x,y\right)\left(1+p^2+q^2\right)^2,$$

and elementary computations show that (1.10) again reduces to the single condition (1.11), i.e. $|\mathcal{K}_2|\leq C\mathcal{K}^{\frac{3}{2}}$.

We observe that the *a priori* estimates $u_{xx},u_{yy}\leq C$ in [6] and [7] for convex solutions u of (1.9) translate into the following estimates on w under the partial Legendre transform:

$$\begin{cases}w_y\left(x,y\right) & \geq \frac{1}{C}\\ 1+w_x\left(x,y\right)^2 & \leq Cw_y\left(x,y\right)\\ k\left(x,w\left(x,y\right)\right)w_y\left(x,y\right) & \leq C\end{cases}.\tag{1.12}$$

Indeed, reverting to the original variables (s,t), the inequalities follow immediately from the *a priori* estimates $u_{xx}\leq C$, $u_{yy}\leq C$ since $\max\left\{k,u_{xy}^2\right\}\leq k+u_{xy}^2=u_{xx}u_{yy}$:

$$\begin{aligned}y_t &= \frac{1}{u_{yy}}\geq\frac{1}{C},\\ k\left(s,y\left(s,t\right)\right)y_t\left(s,t\right) &= k\left(x,y\right)\frac{1}{u_{yy}\left(x,y\right)}\leq u_{xx}\left(x,y\right)\leq C,\\ y_s\left(s,t\right)^2 &= \frac{u_{xy}\left(x,y\right)^2}{u_{yy}\left(x,y\right)^2}\leq\frac{u_{xx}\left(x,y\right)}{u_{yy}\left(x,y\right)}\leq Cy_t\left(s,t\right),\\ k\left(s,y\left(s,t\right)\right)y_s\left(s,t\right)^2 &= k\left(x,y\right)\frac{u_{xy}\left(x,y\right)^2}{u_{yy}\left(x,y\right)^2}\leq u_{xx}\left(x,y\right)^2\leq C^2.\end{aligned}$$

[1] In a paper [10] in preparation with C. Rios, it is shown that the exponent $\frac{3}{2}$ in (1.10) and (1.11) can be replaced with the near optimal exponent $1+\varepsilon$, $\varepsilon>0$.

Note that the fourth inequality also follows by combining the second and third inequalities. We will heretofore assume that all of our solutions to (1.8) satisfy (1.12) as well.

Remark 1.3. *Consider the classical case $k = k(x,y)$. Of importance here is the fact that (1.11) $|k_2| \le Ck^{\frac{3}{2}}$ implies the analogue of (1.5) for $k(x, w(x,y))$;*

$$\begin{aligned} \left|\nabla_{(x,y)} k(x,w(x,y))\right| &\le |k_1(x,w(x,y))| + |k_2(x,w(x,y))| \, |\nabla w(x,y)| \\ &\le C\left(\sqrt{k} + k^{\frac{3}{2}} k^{-1}\right) = C\sqrt{k(x,w(x,y))}, \end{aligned} \tag{1.13}$$

independent *of ∇w, since $|\nabla w| \le Ck^{-1}$ for solutions w to (3.2) satisfying (1.12).*

Just as in Remark 1.3 above, the conditions (1.10) on k_i are precisely those which together with (1.12), imply (1.5) for $\widetilde{k}$, namely $\left|\nabla_{(s,t)}\widetilde{k}\right| \le C\sqrt{\widetilde{k}}$. Note that $d(4) = 1$ is less than $d(2) = d(3) = \frac{3}{2}$ since (3.2) yields $|z_s| = kw_t \le C$ and $|z_t| = |w_s| \le Ck^{-\frac{1}{2}}$ by the *a priori* estimates (1.12), and thus the term $k_4(s, w(s,t), r(s,t), z(s,t), t) \left|\nabla_{(s,t)} z\right|$ will be bounded by $\sqrt{k}$ if $|k_4| \le Ck$. This observation is important in our application to the prescribed Gaussian curvature equation. The reason for the special hypothesis on the second derivative k_{55}, and not the others, is that the strong hypotheses on k_2, k_3 and k_4 actually turn out to imply that $|\nabla k_i| \le C\sqrt{k}$ for $2 \le i \le 4$. Since the very last argument in the paper requires $|k_{ij}| \le C\sqrt{k}$ for $2 \le i, j \le 5$, we see that only the bound on k_{55} requires an additional hypothesis.

Theorem 1.3. *Suppose $k(x,y,v,p,q)$ is smooth and nonnegative in a domain $\Omega \times \mathbb{R}^3$, is positive for $x \ne 0$ and satisfies (1.10). Let ζ and $\varkappa$ be smooth cutoff functions supported in Ω' as above. Then for every multi-index α, there is an increasing real-valued function $\mathcal{C}_\alpha(L)$, defined for $L \in \mathcal{P}_c(\Omega \times \mathbb{R}^3)$, depending only on Ω, Ω', $\sum_{|\beta| \le |\alpha|+2} \left(\left\|D^\beta \zeta\right\|_\infty + \left\|D^\beta \varkappa\right\|_\infty\right)$, $\inf_{\{(x,y,v,p,q)\in L : |x| \ge \varepsilon_\alpha\}} k$ and $\sum_{|\beta| \le |\alpha|+2} \left\|D^\beta k\right\|_{L^\infty(L)}$ where*

$$\varepsilon_\alpha = \varepsilon\left(\Omega, \|k\|_{C^{|\alpha|+2}(L)}, \left\|\frac{|k_1| + |k_5| + |k_{55}|}{\sqrt{k}} + \sum_{i=2}^{4} \frac{|k_i|}{k^{d(i)}}\right\|_{L^\infty(L)}\right) > 0,$$

such that

$$\|\zeta D^\alpha w\|_\infty \le \mathcal{C}_\alpha(L)$$

for all smooth solutions w, z and r of (1.8) in Ω' satisfying (1.12), and such that

$$(x, w(x,y), r(x,y), z(x,y), y) \in L$$

for all (x,y) in the support of $\varkappa$.

In [12] an example is given to show that under the hypotheses of Theorem 1.1, the stronger estimate $\|\zeta D^\alpha w\|_\infty \le \mathcal{C}_\alpha(L)$ in the conclusion of Theorem 1.3 may fail.

Throughout this paper we will use C to denote a constant that may change from line to line, but is independent of any significant quantities. We will use a calligraphic $\mathcal{C}$ to denote a function of one or more variables, increasing in each variable separately, that may also change from line to line, but remains independent of any significant quantities apart from its variables. We will use nonnegative cutoff functions adapted to our operator $\mathcal{L}$ as follows. Let $\mathcal{R} = [-R_1, R_1] \times [-R_2, R_2]$ be a rectangle centred at the origin in the plane, which we assume lies in Ω', and let $\eta_i, \zeta_i, \theta_i \in C_c^\infty((-R_i, R_i))$ for $i = 1, 2$ satisfy

1. η_i equals 1 in a neighbourhood of zero,
2. ζ_i equals 1 in a neighbourhood of zero,
3. $\theta_i = 1$ on the supports of both η_i' and ζ_i',
4. 0 does not lie in the support of θ_i.

Set

$$\begin{aligned} \eta(x,y) &= \eta_1(x)\,\eta_2(y), \\ \zeta(x,y) &= \zeta_1(x)\,\zeta_2(y), \\ \varrho_1(x,y) &= \theta_1(x)\,\zeta_2(y), \\ \varrho_2(x,y) &= \zeta_1(x)\,\theta_2(y). \end{aligned}$$

Let $\xi, \varkappa \in C_c^\infty(\mathcal{R})$ satisfy

1. $\xi = 1$ on the support of all four functions η, ζ, ϱ_1 and ϱ_2,
2. $\varkappa = 1$ on the support of ξ.

Convention: We now introduce a small abuse of notation in order to greatly relieve congestion in subsequent complicated formulas. Many of our quasilinear equations involve functions of the form $(D^\alpha k)(x, w(x,y))$ for a multiindex α. We should of course write this as $(D^\alpha k) \circ \Phi$ where $\Phi(x,y) = (x, w(x,y))$, but will instead write simply $D^\alpha k$ when it is clear that the derivative is evaluated at $\Phi(x,y)$. For example, using the standard notation that k_i denotes partial differentiation of $k(x,y)$ with respect to x if $i = 1$, and y if $i = 2$, we will write k_i and k_{ij} to mean $k_i(x, w(x,y))$ and $k_{ij}(x, w(x,y))$ respectively. In these circumstances, the meaning of the formula $\partial_x k = k_1 + k_2 w_y$ is

$$\partial_x \{k(x, w(x,y))\} = k_1(x, w(x,y)) + k_2(x, w(x,y))\, w_y(x,y).$$

We remark that throughout section 2 on linear equations, k always means $k(x,y)$, while in section 3 on quasilinear equations, k always means $k(x, w(x,y))$. When there is the possibility of confusion, we will write out $k(x,y)$ or $k(x, w(x,y))$ explicitly.

2. Hypoellipticity of Linear Equations

In this section we review the analogous linear theory of hypoelliptic *a priori* estimates, which will be used as part of our attack in the nonlinear case. We denote by $\|f\|_s$ the usual Sobolev space norm given by

$$\|f\|_s^2 = \int_{\mathbb{R}^2} \left|\widehat{f}(\xi)\right|^2 \left(1 + |\xi|^2\right)^{\frac{s}{2}} d\xi.$$

One may obtain the hypoellipticity of the linear operator

$$\mathcal{L} = \partial_x^2 + \partial_y k(x,y)\, \partial_y \tag{2.1}$$

in the case that $k \in C^\infty(\mathcal{R})$, where $\mathcal{R}$ is a rectangle $[-R_1, R_1] \times [-R_2, R_2]$, by first establishing an *a priori* inequality of the form

$$\|\eta u\|_s^2 \leq C\, \|\eta \mathcal{L} u\|_s^2 + C\, \|\xi \mathcal{L} u\|_{s-\varepsilon}^2 + C\, \|\varkappa u\|_{s-\varepsilon}^2, \qquad \varepsilon > 0, \tag{2.2}$$

where the cutoff functions η, ξ and $\varkappa$ are adapted to the operator $\mathcal{L}$ as above. Note that the cutoff function η is replaced by a larger cutoff ξ, which is 1 on the support of η, when $\mathcal{L}u$ is measured in a Sobolev space of smaller order. This is important in deducing the general case from the special case $s=\varepsilon$ by bootstrapping. Note moreover that (2.2) is weaker than subellipticity: the operator $\mathcal{L}$ is subelliptic if there is $\varepsilon>0$ such that

$$\|u\|_{\varepsilon}^{2}\leq C\left(\left|\int(\mathcal{L}u)(u)\right|+\|u\|_{0}^{2}\right),$$

for all smooth compactly supported u. Since the function $\widetilde{k}(x,y)=k(x,w(x,y))$ arising in the quasilinear equation has bounds on its derivatives depending on those of the solution w, we will restrict attention to the case $s=\epsilon=1$,

$$\|\eta u\|_{1}^{2}\leq C\|\eta\mathcal{L}u\|_{1}^{2}+C\|\xi\mathcal{L}u\|_{0}^{2}+C\|\varkappa u\|_{0}^{2},$$

in order to avoid difficult remainder terms arising from the pseudodifferential calculus when s is not integral.

The basic idea of the proof, following J. Kohn [8], is to estimate $\|\eta u\|_{1}^{2}$ by the Poincaré inequality in the x - variable (which requires no information on the degenerate function k),

$$\|\eta u\|_{1}^{2}=\|\nabla\eta u\|_{0}^{2}\leq CR_{1}^{2}\|\partial_{x}(\nabla\eta u)\|_{0}^{2}, \tag{2.3}$$

and then use the k - gradient estimate (compare Corollary 2.3 below),

$$\begin{aligned}\int_{\mathcal{R}}\left(|\zeta\partial_{x}\nabla\eta u|^{2}+k|\zeta\partial_{y}\nabla\eta u|^{2}\right) &\leq -2\int_{\mathcal{R}}(\zeta\mathcal{L}\nabla\eta u)\cdot(\zeta\nabla\eta u)\\ &\quad+4\int_{\mathcal{R}}|\zeta_{x}\nabla\eta u|^{2}+4\int_{\mathcal{R}}k\left|\zeta_{y}\nabla\eta u\right|^{2}\\ &= I+II+III,\end{aligned}$$

exploiting the fact that R_1 is small. In the subelliptic case, where k vanishes to finite order in x, there are Poincaré inequalities that actually improve the L^p integrability of solutions. These are not available here, and the small constant is our only improvement. Term I is handled by writing

$$[\mathcal{L},\nabla\eta]=\nabla[\mathcal{L},\eta]+[\mathcal{L},\nabla]\eta$$

and estimating the commutator $[\mathcal{L},\eta]$ with the help of even and odd operators (see Lemma 2.4 below). The commutator $[\mathcal{L},\nabla]=-\partial_{y}(\nabla k)\partial_{y}$ can be suitably estimated using inequality (1.5), $|\nabla k|\leq C\sqrt{k}$. In both cases, terms of the form $C\|\eta u\|_{1}^{2}$ arise in the estimates, but can be absorbed into the left side of (2.3) since they are multiplied by R_1, which we can take sufficiently small. We remark that $[\mathcal{L},\nabla]$ has no remainder term while $[\mathcal{L},\Lambda^{s}]$ for s not an integer, has a remainder that requires too much smoothness of $k(x,w(x,y))$. Term II is supported where $\mathcal{L}$ is better behaved, actually elliptic by hypothesis, and term III can be handled by exploiting the fact that the weight k in the norm is the least eigenvalue of the operator $\mathcal{L}$; this permits us to replace the identity $v=\nabla\cdot I_{1}v$, $I_{1}=\nabla\cdot\triangle^{-1}$, with the pointwise inequality

$$k|v|^{2}\leq|\partial_{x}I_{1}v|^{2}+k|\partial_{y}I_{1}v|^{2}.$$

We then turn to the L^p estimates of J. Moser, and establish *a priori* inequalities with an improvement in the integrability of derivatives of the solution, similar to

$$\left(\int_{\mathcal{R}}\left|\zeta D^{\alpha}u^{\beta}\right|^{2}dxdy\right)^{\frac{1}{2}}\leq C\beta\left(\int_{\mathcal{R}}\left|\xi D^{\alpha}u^{\beta}\right|^{p}dxdy\right)^{\frac{1}{p}}, \tag{2.4}$$

for some $p < 2$. This will be useful in estimating the nonlinearities in $\mathcal{L}u$ in the quasilinear case. Next, we consider the quasilinear degenerate elliptic equation (1.1),

$$\mathcal{L}w = \left[\partial_x^2 + \partial_y k\left(x, w\left(x, y\right)\right)\partial_y\right] w = 0,$$

where k is smooth and nonnegative on $\mathcal{R}$ and w is smooth. We alternately apply the *a priori* inequalities (2.3) and (2.4) to obtain that the derivatives of w are controlled by $\|w\|_\infty$ and $\|\nabla w\|_\infty$.

Finally, it might be helpful to keep the following points in mind while reading the estimates in subsequent sections. Since u is a solution of $\mathcal{L}u = 0$, the operator $\mathcal{L}$ behaves better than an operator of order 2 when applied to u. However, when $\mathcal{L}$ is commuted with an operator P of order α, then $\mathcal{L}$ loses its special status in $[\mathcal{L}, P]$, and the commutator has order only $2 + \alpha - 1$. In order to compensate for this loss, we need to exploit special properties of $[\mathcal{L}, P]$: the inequality (1.5) in case P is a differential operator of order $\alpha = 1$, the even-odd technology in case P is multiplication by a cutoff function with $\alpha = 0$, and the pseudodifferential calculus of rough operators in the case P has order $\alpha = -1$ (see the proof of Lemma 2.10 below).

2.1. **The gradient estimate.** Let k be nonnegative and smooth on $\mathcal{R}$ (we remind the reader that throughout this section $k = k\left(x, y\right)$). We begin with the well known Caccioppoli inequality estimating the energy of the $\mathcal{L}$ - gradient of a function u in terms of u and $\mathcal{L}u$. For this it is convenient to introduce the inner product

$$\left\langle v, w\right\rangle_k = v_1 w_1 + k v_2 w_2 = v_1 w_1 + k\left(x, y\right) v_2 w_2,$$

as well as the matrix

$$\mathcal{A} = \mathcal{A}\left(x, y\right) = \begin{bmatrix} 1 & 0 \\ 0 & k\left(x, y\right) \end{bmatrix},$$

so that $\mathcal{L} = \nabla \cdot \mathcal{A} \nabla$. The operator $\mathcal{L}$ is a sum of "squares" of the two vector fields ∂_x and $\sqrt{k}\partial_y$, usually called the unit vector fields associated with $\mathcal{L}$. Later it will be important to observe that the vector fields $k_i \partial_y$ are subunit in the sense that $|k_i| \leq C\sqrt{k}$ by (1.5).

Lemma 2.1. *Suppose $\mathcal{L}$ is as in (2.1) with k nonnegative and smooth. For $u \in C^\infty\left(\mathcal{R}\right)$, we have the identity*

$$\int_{\mathcal{R}} \left\langle \zeta \nabla u, \zeta \nabla u\right\rangle_k dxdy = -\int_{\mathcal{R}} \left(\zeta \mathcal{L}u\right)\left(\zeta u\right) dxdy - 2\int_{\mathcal{R}} \left\langle u \nabla \zeta, \zeta \nabla u\right\rangle_k dxdy,$$

and the inequality

$$\int_{\mathcal{R}} \left(\left|\zeta \partial_x u\right|^2 + k\left|\zeta \partial_y u\right|^2\right) dxdy \leq -2\int_{\mathcal{R}} \left(\zeta \mathcal{L}u\right)\left(\zeta u\right) dxdy \tag{2.5}$$
$$+4\left\|\zeta_x\right\|_\infty^2 \int_{\mathcal{R}} \left|\varrho_1 u\right|^2 dxdy + 4\left\|\zeta_y\right\|_\infty^2 \int_{\mathcal{R}} k\left|\varrho_2 u\right|^2 dxdy.$$

Proof. Integration by parts yields the identity above, and then using

$$2\left|\left\langle u \nabla \zeta, \zeta \nabla u\right\rangle_k\right| \leq \frac{1}{2}\left\langle \zeta \nabla u, \zeta \nabla u\right\rangle_k + 2\left\langle u \nabla \zeta, u \nabla \zeta\right\rangle_k$$

in the identity, and absorbing the term reproduced on the right, yields

$$\int_{\mathcal{R}} \left\langle \zeta \nabla u, \zeta \nabla u\right\rangle_k \leq -2\int_{\mathcal{R}} \left(\zeta \mathcal{L}u\right)\left(\zeta u\right) + 4\int_{\mathcal{R}} \left\langle u \nabla \zeta, u \nabla \zeta\right\rangle_k.$$

Inequality (2.5) now follows from $\langle \nabla\zeta, \nabla\zeta\rangle_k = \zeta_x^2 + k\zeta_y^2 \le \|\zeta_x\|_\infty^2 \rho_1^2 + \left\|\zeta_y\right\|_\infty^2 k\rho_2^2$.

Corollary 2.2. *Suppose $\mathcal{L}$ is as in (2.1) with k nonnegative and smooth. For $u \in C^\infty(\mathcal{R})$, we have*

$$\int_{\mathcal{R}} \left(|\partial_x \zeta u|^2 + k\left|\partial_y \zeta u\right|^2\right) dxdy \le -4\int_{\mathcal{R}} (\zeta\mathcal{L}u)(\zeta u)\, dxdy + 10\|\zeta_x\|_\infty^2 \int_{\mathcal{R}} |\varrho_1 u|^2 dxdy + 10\left\|\zeta_y\right\|_\infty^2 \int_{\mathcal{R}} k|\varrho_2 u|^2 dxdy. \tag{2.6}$$

Proof. Use $\partial_x\zeta u = \zeta\partial_x u + \zeta_x u$, $\partial_y \zeta u = \zeta\partial_y u + \zeta_y u$ and (2.5).

2.1.1. *Gradients and commutators.* In this subsubsection, we extend the k-gradient estimate for ζu in (2.6) to a k-gradient estimate for a deriviative $\zeta\partial\eta u$. It will be convenient to set

$$A^6 = 1 + \|\nabla\eta\|_\infty^6 + \|\nabla\zeta\|_\infty^6 + \|\nabla\varrho_1\|_\infty^6 + \|\nabla\varrho_2\|_\infty^6 + \left\|\nabla^2\eta\right\|_\infty^3 + \left\|\nabla^2\zeta\right\|_\infty^3 + \left\|\nabla^3\eta\right\|_\infty^2, \tag{2.7}$$

in order to collect constants in front of the lower order terms in what follows. It is important to observe that since $A \ge R_1^{-1}$, if we wish to show that a certain term is small by applying the one-dimensional Poincaré inequality in the x-variable in order to gain a factor of R_1 (as in (2.3) above), we must ensure that the term to be shown small is not multiplied by a constant which increases with A.

Corollary 2.3. *Suppose $\mathcal{L}$ is as in (2.1) with k nonnegative and smooth. Let ∂ denote either ∂_x or ∂_y. For $u \in C^\infty(\mathcal{R})$, we have*

$$\begin{aligned} &\int_{\mathcal{R}} \left(|\partial_x(\zeta\partial\eta u)|^2 + k\left|\partial_y(\zeta\partial\eta u)\right|^2\right) dxdy \\ \le\ & -4\int_{\mathcal{R}} (\zeta\partial\eta\mathcal{L}u)(\zeta\partial\eta u)\, dxdy - 4\int_{\mathcal{R}} (\zeta[\mathcal{L},\partial\eta]u)(\zeta\partial\eta u)\, dxdy \\ & + CA^2\|\varrho_1 u\|_1^2 + CA^2\left\|\sqrt{k}\partial\varrho_2 u\right\|_0^2 + CA^4\|\xi u\|_0^2. \end{aligned} \tag{2.8}$$

Proof. Replace u with $\partial\eta u$ in (2.6) and then use $\mathcal{L}\partial\eta = \partial\eta\mathcal{L} + [\mathcal{L},\partial\eta]$ and $\varrho_i\partial\eta = [\varrho_i,\partial]\eta + [\partial,\eta]\varrho_i + \eta\partial\varrho_i$ for $i = 1, 2$.

The above corollary leads us to consideration of the commutator appearing on the right side of (2.8),

$$[\mathcal{L},\partial\eta] = \partial[\mathcal{L},\eta] + [\mathcal{L},\partial]\eta. \tag{2.9}$$

The two terms on the right side of (2.9) will be estimated in Lemmas 2.5 and 2.6 below. In order to handle the commutator $[\mathcal{L},\eta]$, we will need a standard lemma regarding even and odd operators (see e.g. [8]). Let ∂ denote a partial derivative of order one, either ∂_x or ∂_y, so that $\partial + \partial^t = 0$.

Lemma 2.4. *Let P and Q denote classical pseudodifferential operators such that $P + P^t$ and $Q + Q^t$ have order one less than the order $ord(P)$ and $ord(Q)$ of P and Q respectively. For $u \in C^\infty$, we have the following identity in which the sum of the orders of the operators appearing in the terms on the right is one less than* *the sum on the left:*

$$2\int (P\partial\zeta u)(Q\zeta u) = \int ([P,\partial]\zeta u)(Q\zeta u) + \int (P\zeta u)([Q,\partial]\zeta u) + \int ([P,Q^t]\zeta u)(\partial\zeta u) + \int ((P+P^t)\partial\zeta u)(Q\zeta u) - \int (P^t\partial\zeta u)((Q+Q^t)\zeta u).$$

We can now use inequality (1.5) to handle the right side of (2.8) that involves the first term $\partial[\mathcal{L},\eta]$ in (2.9).

Lemma 2.5. *Suppose $\mathcal{L}$ is as in (2.1) with k nonnegative, smooth and satisfying (1.5). Let ∂ denote either ∂_x or ∂_y. For $u \in C^\infty(\mathcal{R})$ and $0<\alpha<1$, we have with B as in (1.5),*

$$\left|\int_{\mathcal{R}} (\zeta\partial[\mathcal{L},\eta]u)(\zeta\partial\eta u)\,dxdy\right| \le C\alpha\left(B^2+1\right)\left(\|\partial_x\zeta\partial\eta u\|_0^2 + \left\|\sqrt{k}\partial_y\zeta\partial\eta u\right\|_0^2\right) + C\|\eta u\|_1^2 + C\left(A^4\|\varrho_1 u\|_1^2 + \left(A^4+\frac{A^2}{\alpha}\right)\left\|\sqrt{k}\partial_y\varrho_2 u\right\|_0^2\right) + A^4\left(\frac{1}{\alpha}+B^2+A^2\right)\|\xi u\|_0^2 + C\alpha\left(B^2+1\right)A^2\left|\int_{\mathcal{R}} (\eta\mathcal{L}u)(\eta u)\,dxdy\right|.$$

Proof. Computing out $[\mathcal{L},\eta]$ we obtain

$$\begin{aligned}\left|\int_{\mathcal{R}} (\zeta\partial[\mathcal{L},\eta]u)(\zeta\partial\eta u)\right| &\le 2\left|\int_{\mathcal{R}} (\zeta\partial\eta_x\partial_x u)(\zeta\partial\eta u)\right| + 2\left|\int_{\mathcal{R}} (\zeta\partial k\eta_y\partial_y u)(\zeta\partial\eta u)\right| \\ &\quad + \left|\int_{\mathcal{R}} \left(\zeta\partial\left(\eta_{xx}+k\eta_{yy}+k_y\eta_y\right)u\right)(\zeta\partial\eta u)\right| \\ &= I + II + III.\end{aligned}$$

To estimate I, we note that since $\eta_x = \eta_x\varrho_1^2$, we have

$$\begin{aligned}-\int_{\mathcal{R}} (\zeta\partial\eta_x\partial_x u)(\zeta\partial\eta u) &= \int_{\mathcal{R}} \left(\partial\zeta^2\partial\varrho_1\eta_x\varrho_1\partial_x u\right)(\eta u) \\ &= \int_{\mathcal{R}} \left(\varrho_1\partial\zeta^2\partial\eta_x\partial_x\varrho_1 u\right)(\eta u) - \int_{\mathcal{R}} \left(\partial\zeta^2\partial\varrho_1\eta_x(\partial_x\varrho_1)u\right)(\eta u) \\ &\quad - \int_{\mathcal{R}} \left(\left[\varrho_1,\partial\zeta^2\partial\right]\eta_x\partial_x\varrho_1 u\right)(\eta u),\end{aligned}$$

and so, as we shall show,

$$\left|\int_{\mathcal{R}} (\zeta\partial\eta_x\partial_x u)(\zeta\partial\eta u)\right| \le \left|\int_{\mathcal{R}} (\zeta\partial\eta_x\partial_x\varrho_1 u)(\zeta\partial\eta\varrho_1 u)\right| + C\left(\|\eta u\|_1^2 + A^4\|\varrho_1 u\|_1^2 + A^6\|\xi u\|_0^2\right). \tag{2.10}$$

Indeed, the first term on the right side of (2.10) is the absolute value of the first term on the right side of the previous display. We also have

$$\begin{aligned}\left|\int_{\mathcal{R}} \left(\partial\zeta^2\partial\varrho_1\eta_x(\partial_x\varrho_1)u\right)(\eta u)\right| &= \left|\int_{\mathcal{R}} \left(\zeta^2\partial\varrho_1\eta_x(\partial_x\varrho_1)u\right)(\partial\eta u)\right| \\ &\le \left|\int_{\mathcal{R}} \left(\zeta^2\eta_x(\partial_x\varrho_1)\partial\varrho_1 u\right)(\partial\eta u)\right| + \left|\int_{\mathcal{R}} \left(\zeta^2\left[\partial\varrho_1,\eta_x(\partial_x\varrho_1)\right]u\right)(\partial\eta u)\right| \\ &\le C\left(\|\eta u\|_1^2 + A^4\|\varrho_1 u\|_1^2 + A^6\|\xi u\|_0^2\right),\end{aligned}$$

since $[\partial\varrho_1, \eta_x(\partial_x\varrho_1)]$ has order 0 and norm bounded by A^3. Similarly,

$$\left|\int_{\mathcal{R}} \left(\left[\varrho_1, \partial\zeta^2\partial\right]\eta_x\partial_x\varrho_1 u\right)(\eta u)\right| \leq C\left(\|\eta u\|_1^2 + A^4\|\varrho_1 u\|_1^2 + A^6\|\xi u\|_0^2\right)$$

since $\left[\varrho_1, \partial\zeta^2\partial\right]$ is the sum of a zero order operator of norm A^2 and a first order operator of norm A, upon expanding the commutator. Now apply Lemma 2.4, with $P = \zeta\partial\eta_x$ and $Q = \zeta\partial\eta$ to obtain from (2.10) that

$$|I| \leq C\left(\|\eta u\|_1^2 + A^4\|\rho_1 u\|_1^2 + A^6\|\xi u\|_0^2\right).$$

We remark that since $P + P^t = \zeta\partial\eta_x - \eta_x\partial\zeta$ has order 0 (and similarly for $Q + Q^t$), all the terms on the right side of Lemma 2.4 have less total order than the left side, and after much computation we have the desired result. Note also the tradeoff of order for powers of A in Lemma 2.4 - if a derivative hits a cutoff function, the order is reduced but an additional factor of A arises in the norm.

For II we write

$$\begin{aligned} |II| &= 2\left|\int_{\mathcal{R}}\left(\sqrt{k}\eta_y\partial_y u\right)\left(\sqrt{k}\partial\zeta^2\partial\eta u\right)\right| \\ &\leq C\frac{1}{\alpha}\left\|\sqrt{k}\eta_y\partial_y u\right\|_0^2 + C\alpha\left\|\sqrt{k}\partial\zeta^2\partial\eta u\right\|_0^2. \end{aligned}$$

We may assume $\varrho_2 = 1$ on the support of η_y if we assume that $\zeta_1 = 1$ on the support of η_1, since $\varrho_2 = \zeta_1(x)\theta_2(y)$ and $\eta_y = \eta_1(x)\eta_2'(y)$.

Cautionary Note: We initially defined the cutoff functions ζ_i and η_i to be independent for $i = 1, 2$. We caution the reader that while we will now assume that $\zeta_1 = 1$ on the support of η_1, in later sections we will want to choose just the opposite, namely $\eta_i = 1$ on the support of ζ_i. This will not be circular, as in the iterations of our inequalities, we replace our existing complement of cutoff functions with a completely new collection, supported in a much larger set and often without notice.

So with $\varrho_2 = 1$ on the support of η_y we have

$$\left\|\sqrt{k}\eta_y\partial_y u\right\|_0^2 \leq A^2\left\|\sqrt{k}\partial_y\varrho_2 u\right\|_0^2,$$

and

$$\begin{aligned} \left\|\sqrt{k}\partial\zeta^2\partial\eta u\right\|_0^2 &\leq C\left\|\zeta\sqrt{k}\partial\zeta\partial\eta u\right\|_0^2 + C\left\|\sqrt{k}(\partial\zeta)\partial\eta u\right\|_0^2 \\ &\leq C\|\partial_x\zeta\partial\eta u\|_0^2 + C\left\|\sqrt{k}\partial_y\zeta\partial\eta u\right\|_0^2 \\ &\quad + CA^2\|\partial_x\eta u\|_0^2 + CA^2\left\|\sqrt{k}\partial_y\eta u\right\|_0^2 \end{aligned} \tag{2.11}$$

upon considering the cases $\partial = \partial_x$ and $\partial = \partial_y$ separately, throwing away the $\sqrt{k}$ when $\partial = \partial_x$. Thus we obtain

$$\begin{aligned} |II| &\leq C\alpha\|\partial_x\zeta\partial\eta u\|_0^2 + C\alpha\left\|\sqrt{k}\partial_y\zeta\partial\eta u\right\|_0^2 \\ &\quad + C\alpha A^2\|\partial_x\eta u\|_0^2 + C\alpha A^2\left\|\sqrt{k}\partial_y\eta u\right\|_0^2 + C\frac{A^2}{\alpha}\left\|\sqrt{k}\partial_y\varrho_2 u\right\|_0^2. \end{aligned}$$

We now apply Corollary 2.2 to estimate the middle line above by

$$C\alpha A^2 \|\partial_x \eta u\|_0^2 + C\alpha A^2 \left\|\sqrt{k}\partial_y \eta u\right\|_0^2 \leq C\alpha A^2 \left|\int_{\mathcal{R}} (\eta \mathcal{L} u)(\eta u)\right| + C\alpha A^4 \|\xi u\|_0^2 . \tag{2.12}$$

Finally, for III, we have

$$|III| \leq \left|\int_{\mathcal{R}} (\zeta \partial \eta_{xx} u)(\zeta \partial \eta u)\right| + \left|\int_{\mathcal{R}} \left(\zeta \partial k \eta_{yy} u\right)(\zeta \partial \eta u)\right| + \left|\int_{\mathcal{R}} \left(\zeta \partial k_y \eta_y u\right)(\zeta \partial \eta u)\right| .$$

Using $\eta_{xx} = \eta_{xx} \varrho_1$, we see that the first of the three terms is dominated by

$$C\left(\|\eta u\|_1^2 + A^4 \|\varrho_1 u\|_1^2 + A^6 \|\xi u\|_0^2\right).$$

Using $\eta_{yy} = \eta_{yy} \varrho_2$ and our hypothesis (1.5), we see that in the case $\partial = \partial_y$, the second term is dominated by

$$C\left(\|\eta u\|_1^2 + A^4 \|k\partial_y \varrho_2 u\|_0^2 + A^4 \left(A^2 + B^2\right) \|\xi u\|_0^2\right).$$

In the case $\partial = \partial_x$, we have

$$\begin{aligned} \left|\int_{\mathcal{R}} \left(\zeta \partial_x k \eta_{yy} u\right)(\zeta \partial_x \eta u)\right| &= \left|\int_{\mathcal{R}} \left(k \eta_{yy} u\right)\left(\partial_x \zeta^2 \partial_x \eta u\right)\right| \\ &\leq \frac{C}{\alpha} A^4 \|\xi u\|_0^2 + C\alpha \left\|\partial_x \zeta^2 \partial_x \eta u\right\|_0^2 \\ &\leq \frac{C}{\alpha} A^4 \|\xi u\|_0^2 + C\alpha \|\partial_x \zeta \partial_x \eta u\|_0^2 + C\alpha A^2 \|\partial_x \eta u\|_0^2 , \end{aligned}$$

and we can apply (2.12) to the last term here. Finally, by using (2.11) and our hypothesis (1.5), the third term satisfies

$$\begin{aligned} \left|\int_{\mathcal{R}} \left(\zeta \partial k_2 \eta_y u\right)(\zeta \partial \eta u)\right| &= \left|\int_{\mathcal{R}} \left(\eta_y u\right)\left(k_2 \partial \zeta^2 \partial \eta u\right)\right| \\ &\leq \frac{C}{\alpha} A^2 \|\xi u\|_0^2 + C\alpha B^2 \left\|\sqrt{k}\partial \zeta^2 \partial \eta u\right\|_0^2 \\ &\leq \frac{C}{\alpha} A^2 \|\xi u\|_0^2 + B^2 C\alpha \|\partial_x \zeta \partial \eta u\|_0^2 + B^2 C\alpha \left\|\sqrt{k}\partial_y \zeta \partial \eta u\right\|_0^2 \\ &\quad + C\alpha A^2 B^2 \|\partial_x \eta u\|_0^2 + C\alpha A^2 B^2 \left\|\sqrt{k}\partial_y \eta u\right\|_0^2 . \end{aligned}$$

Now use (2.12) on the final two terms on the right side to complete the proof of Lemma 2.5.

We can now use inequality (1.5) to handle the right side of (2.8) that involves the second term $[\mathcal{L}, \partial] \eta$ in (2.9).

Lemma 2.6. *Suppose $\mathcal{L}$ is as in (2.1) with k nonnegative, smooth and satisfying (1.5). Let ∂ denote either ∂_x or ∂_y. For $u \in C^\infty(\mathcal{R})$ and $0 < \alpha < 1$, we have*

$$\begin{aligned} \left|\int_{\mathcal{R}} (\zeta [\mathcal{L}, \partial] \eta u)(\zeta \partial \eta u)\, dxdy\right| &\leq C\frac{1}{\alpha} \|\eta u\|_1^2 + CB^2 \alpha \int_{\mathcal{R}} k \left|\partial_y \zeta \partial \eta u\right|^2 dxdy \\ &\quad + C\alpha A^2 B^2 \left|\int_{\mathcal{R}} (\eta \mathcal{L} u)(\eta u)\, dxdy\right| + C\alpha A^4 B^2 \|\xi u\|_0^2 . \end{aligned}$$

Proof. Use $[\mathcal{L},\partial]=\partial_y[k,\partial]\partial_y=-\partial_y(\partial k)\partial_y$ along with (1.5) and (2.12).

We can now replace the right side of (2.8) with only error terms and terms involving $\mathcal{L}u$.

Corollary 2.7. *Suppose $\mathcal{L}$ is as in (2.1) with k nonnegative, smooth and satisfying (1.5). Let ∂ denote either ∂_x or ∂_y. For $u\in C^\infty(\mathcal{R})$, we have*

$$\begin{aligned}
&\int_{\mathcal{R}}\left(|\partial_x(\zeta\partial\eta u)|^2+k|\partial_y(\zeta\partial\eta u)|^2\right)dxdy\\
\leq\quad&-4\int_{\mathcal{R}}(\zeta\partial\eta\mathcal{L}u)(\zeta\partial\eta u)\,dxdy+CA^2\left|\int_{\mathcal{R}}(\eta\mathcal{L}u)(\eta u)\,dxdy\right|\\
&+CA^4\|\varrho_1u\|_1^2+CA^2\left(A^2+B^2\right)\int_{\mathcal{R}}k|\partial_y\varrho_2u|^2\,dxdy+CA^2\int_{\mathcal{R}}k|\partial\varrho_2u|^2\,dxdy\\
&+C\left(1+B^2\right)\|\eta u\|_1^2+CA^4\left(A^2+B^2\right)\|\xi u\|_0^2.
\end{aligned}$$

Proof. We plug the identity $[\mathcal{L},\partial\eta]=\partial[\mathcal{L},\eta]+[\mathcal{L},\partial]\eta$ into the second term on the right side of Corollary 2.3,

$$\begin{aligned}
&\int_{\mathcal{R}}\left(|\partial_x(\zeta\partial\eta u)|^2+k|\partial_y(\zeta\partial\eta u)|^2\right)\\
\leq\quad&-4\int_{\mathcal{R}}(\zeta\partial\eta\mathcal{L}u)(\zeta\partial\eta u)-4\int_{\mathcal{R}}(\zeta[\mathcal{L},\partial\eta]u)(\zeta\partial\eta u)\\
&+CA^2\|\varrho_1u\|_1^2+CA^2\left\|\sqrt{k}\partial\varrho_2u\right\|_0^2+CA^4\|\xi u\|_0^2,
\end{aligned}$$

and then estimate the resulting terms with Lemma 2.5 for $0<\alpha<1$ to be chosen,

$$\begin{aligned}
&\left|\int_{\mathcal{R}}(\zeta\partial[\mathcal{L},\eta]u)(\zeta\partial\eta u)\right|\\
\leq\quad&C\alpha\left(B^2+1\right)\left(\|\partial_x\zeta\partial\eta u\|_0^2+\left\|\sqrt{k}\partial_y\zeta\partial\eta u\right\|_0^2\right)+C\|\eta u\|_1^2\\
&+C\left(A^4\|\varrho_1u\|_1^2+\left(A^4+\frac{A^2}{\alpha}\right)\left(\left\|\sqrt{k}\partial_y\varrho_2u\right\|_0^2\right)+A^4\left(\frac{1}{\alpha}+B^2+A^2\right)\|\xi u\|_0^2\right)\\
&+C\alpha A^2\left(B^2+1\right)\left|\int_{\mathcal{R}}(\eta\mathcal{L}u)(\eta u)\right|,
\end{aligned}$$

and Lemma 2.6,

$$\begin{aligned}
\left|\int_{\mathcal{R}}(\zeta[\mathcal{L},\partial]\eta u)(\zeta\partial\eta u)\right|\quad\leq\quad&C\frac{1}{\alpha}\|\eta u\|_1^2+CB^2\alpha\int_{\mathcal{R}}k|\partial_y\zeta\partial\eta u|^2\\
&+C\alpha A^2B^2\left|\int_{\mathcal{R}}(\eta\mathcal{L}u)(\eta u)\right|+C\alpha A^4B^2\|\xi u\|_0^2.
\end{aligned}$$

Then choose $\alpha=\frac{1}{2C(1+B^2)}$ so that the term

$$C\alpha\left(1+B^2\right)\|\partial_x\zeta\partial\eta u\|_0^2+C\alpha\left(1+B^2\right)\int_{\mathcal{R}}k|\partial_y\zeta\partial\eta u|^2$$

can be absorbed into the left side.

2.2. **The Moser Iteration.** In this section we establish local L^p improvement for solutions u of $\mathcal{L}u = 0$, where $\mathcal{L} = \partial_x^2 + \partial_y k(x,y)\,\partial_y$. Whenever we use β to denote a positive real number, we assume that $\beta = \frac{m}{n}$ is rational with n odd, so that expressions such as $u(x)^\beta$ make sense. Let $\mathcal{R} = [-R_1, R_1] \times [-R_2, R_2]$ be a rectangle in the plane, and let $\eta, \zeta, \varrho, \xi, \varkappa$ be as in section 1. Let k be nonnegative and smooth in a neighbourhood of $\mathcal{R}$.

2.2.1. *The gradient estimate for powers.* We begin by generalizing Lemma 2.1 to powers of u as in [9]. Recall that $\langle v, w\rangle_k = v_1 w_1 + k v_2 w_2$ and $\mathcal{A} = \begin{bmatrix} 1 & 0 \\ 0 & k(x,y) \end{bmatrix}$.

Lemma 2.8. *Suppose $\mathcal{L}$ is as in (2.1) with k nonnegative and smooth. For $u \in C^\infty(\mathcal{R})$ and $\beta > \frac{1}{2}$, we have*

$$\int_{\mathcal{R}} \left\langle \zeta \nabla u^\beta, \zeta \nabla u^\beta \right\rangle_k dxdy = \\ -\frac{\beta^2}{2\beta - 1} \int_{\mathcal{R}} (\zeta \mathcal{L} u)\left(\zeta u^{2\beta-1}\right) dxdy - \frac{2\beta}{2\beta-1} \int_{\mathcal{R}} \left\langle u^\beta \nabla \zeta, \zeta \nabla u^\beta \right\rangle_k dxdy,$$

and

$$\int_{\mathcal{R}} \left(\left|\zeta \partial_x u^\beta\right|^2 + k \left|\zeta \partial_y u^\beta\right|^2 \right) dxdy \leq \frac{2\beta^2}{2\beta-1} \left| \int_{\mathcal{R}} (\zeta \mathcal{L} u)\left(\zeta u^{2\beta-1}\right) dxdy \right| \\ + \left(\frac{2\beta}{2\beta-1}\right)^2 \|\zeta_x\|_\infty^2 \int_{\mathcal{R}} \left|\varrho_1 u^\beta\right|^2 dxdy + \left(\frac{2\beta}{2\beta-1}\right)^2 \left\|\zeta_y\right\|_\infty^2 \int_{\mathcal{R}} k \left|\varrho_2 u^\beta\right|^2 dxdy.$$

Proof. For $\beta \geq 1$, integration by parts yields the identity above, and for $\frac{1}{2} < \beta < 1$, an additional elementary limiting argument is needed, which we omit. Now use the inequality $2ab \leq \alpha a^2 + \frac{1}{\alpha} b^2$ to obtain

$$2 \left| \left\langle u^\beta \nabla \zeta, \zeta \nabla u^\beta \right\rangle_k \right| \leq \frac{2\beta - 1}{2\beta} \left\langle \zeta \nabla u^\beta, \zeta \nabla u^\beta \right\rangle_k + \frac{2\beta}{2\beta-1} \left\langle u^\beta \nabla \zeta, u^\beta \nabla \zeta \right\rangle_k,$$

and combining this with $\langle \nabla\zeta, \nabla\zeta\rangle_k = \zeta_x^2 + k\zeta_y^2 \leq \|\zeta_x\|_\infty^2 \rho_1^2 + \|\zeta_x\|_\infty^2 k\rho_2^2$ as in Lemma 2.1, we obtain the desired inequality.

2.2.2. *The subunit estimate.* While the integral $\int_{\mathcal{R}} \left|\varrho_1 u^\beta\right|^2$ in Lemma 2.8 can be handled since it is supported where $\mathcal{L}$ is elliptic, the integral $\int_{\mathcal{R}} k \left|\varrho_2 u^\beta\right|^2$ requires further work. We will use the following fractional integral result repeatedly in this effort.

Proposition 2.9. *Suppose T is a pseudodifferential operator of order $\alpha \in (-2, 0]$. Then*

$$\|\zeta T \xi f\|_{L^q(\mathbb{R}^2)} \leq C \|\xi f\|_{L^p(\mathbb{R}^2)}, \qquad \frac{1}{q} \geq \frac{1}{p} + \frac{\alpha}{2},$$

provided $1 \leq p \leq q < \infty$, and $q < \frac{2}{2+\alpha}$ in the case $p = 1$. If T is in addition a Fourier multiplier operator, then the cutoff function ξ can be omitted.

We do not need (1.5), the inequality $|\nabla k| \leq B\sqrt{k}$, for the next result.

Lemma 2.10. *Suppose $\mathcal{L}$ is as in (2.1) with k nonnegative and smooth. Then for each $\nu > 0$, there is $p < 2$ such that for all $u \in C_c^\infty(\mathcal{R})$ and all $\beta > 1$, $\left(\int_{\mathcal{R}} k\left|\varrho_2 u^\beta\right|^2 dxdy\right)^{\frac{1}{2}}$ is dominated by*

$$C\sqrt{\beta}\left|\int_{\mathcal{R}}\left(\xi I_1\varrho_2 u^{\beta-1}\mathcal{L}u\right)\left(\xi I_1\varrho_2 u^\beta\right)\right|^{\frac{1}{2}} + C\left(\frac{\beta}{\beta-1}\right)^{\frac{1}{2}}\left|\int_{\mathcal{R}}\left(\sqrt{\varrho_2}u^{\frac{\beta}{2}-1}\mathcal{L}u\right)\left(\sqrt{\varrho_2}u^{\frac{\beta}{2}}\right)\right|$$

$$+C_p\left(\sqrt{\beta(\beta-1)} + \frac{A^2\sqrt{\beta}}{(\beta-1)^{\frac{3}{2}}} + \|\varkappa k\|_{C^\nu} + A\|\varkappa k_y\|_\infty + A^2\right)\left(\int_{\mathcal{R}}\left|\xi u^\beta\right|^p\right)^{\frac{1}{p}},$$

where I_1 is a Fourier multiplier operator of order -1.

Proof. Denote by Λ^s the multiplier operator with symbol $\left(1+|\cdot|^2\right)^{\frac{s}{2}}$. We use the identity,

$$Id = \left(I-\nabla^2\right)\Lambda^{-2} = \Lambda^{-2} - \nabla\cdot\left(\nabla\Lambda^{-2}\right),$$

to write

$$\begin{aligned}\int_{\mathcal{R}} k\left|\varrho_2 u^\beta\right|^2 &= \int_{\mathcal{R}} k\left|\xi\varrho_2 u^\beta\right|^2 \le C\int_{\mathcal{R}} k\left|\xi\Lambda^{-2}\varrho_2 u^\beta\right|^2 + C\int_{\mathcal{R}} k\left|\xi\nabla\cdot\left(\nabla\Lambda^{-2}\right)\varrho_2 u^\beta\right|^2\\ &\le C\int_{\mathcal{R}}\left|\xi\Lambda^{-2}\varrho_2 u^\beta\right|^2 + C\int_{\mathcal{R}}\left|\xi\partial_x\left(I_1\varrho_2 u^\beta\right)\right|^2 + C\int_{\mathcal{R}} k\left|\xi\partial_y\left(I_1\varrho_2 u^\beta\right)\right|^2\end{aligned}$$

where $I_1 = \partial_x\Lambda^{-2}$ in the second integral on the right, and $I_1 = \partial_y\Lambda^{-2}$ in the third integral. Both operators I_1 have order -1, and this small abuse of notation should cause no problems. Now the first term on the right satisfies

$$\int_{\mathcal{R}}\left|\xi\Lambda^{-2}\varrho_2 u^\beta\right|^2 \le C_p\left(\int_{\mathcal{R}}\left|\xi u^\beta\right|^p\right)^{\frac{2}{p}}$$

for any $1 \le p < 2$, by Proposition 2.9 on fractional integration (Λ^{-2} has order α for all $\alpha > -2$). By Lemma 2.1, and with $\mathcal{L}$ as in (2.1), the last two terms on the right are dominated by

$$C\left|\int_{\mathcal{R}}\left(\xi\mathcal{L}I_1\varrho_2 u^\beta\right)\left(\xi I_1\varrho_2 u^\beta\right)\right| + CA^2\int_{\mathcal{R}}\left|\varkappa I_1\varrho_2 u^\beta\right|^2 \tag{2.13}$$

upon replacing u by $I_1\varrho_2 u^\beta$ in (2.5). Strictly speaking, we should replace I_1 in (2.13) by $\partial_x\Lambda^{-2}$, and then by $\partial_y\Lambda^{-2}$, and finally add the two expressions. Now the last term in (2.13) satisfies

$$\int_{\mathcal{R}}\left|I_1\varrho_2 u^\beta\right|^2 \le C_p\left(\int_{\mathcal{R}}\left|\xi u^\beta\right|^p\right)^{\frac{2}{p}}$$

for any $1 < p < 2$, by Proposition 2.9 again. It remains to estimate the first term on the right side of (2.13) given by

$$\int_{\mathcal{R}}\left(\xi\mathcal{L}I_1\varrho_2 u^\beta\right)\left(\xi I_1\varrho_2 u^\beta\right) = \int_{\mathcal{R}}\left(\xi I_1\varrho_2\mathcal{L}u^\beta\right)\left(\xi I_1\varrho_2 u^\beta\right) + \int_{\mathcal{R}}\left(\xi\left[\mathcal{L}, I_1\varrho_2\right]u^\beta\right)\left(\xi I_1\varrho_2 u^\beta\right).$$

Noting that

$$\mathcal{L}u^\beta = \left(\partial_x^2 + \partial_y k\partial_y\right)u^\beta = \beta u^{\beta-1}\mathcal{L}u + \beta(\beta-1)u^{\beta-2}\left(|\partial_x u|^2 + k|\partial_y u|^2\right),$$

we have

$$\begin{aligned}
&\int_{\mathcal{R}} \left(\xi \mathcal{L} I_1 \varrho_2 u^{\beta}\right)\left(\xi I_1 \varrho_2 u^{\beta}\right) = \beta \int_{\mathcal{R}} \left(\xi I_1 \varrho_2 u^{\beta-1} \mathcal{L} u\right)\left(\xi I_1 \varrho_2 u^{\beta}\right) \\
&\quad + \beta(\beta-1) \int_{\mathcal{R}} \left(\xi I_1 \varrho_2 u^{\beta-2}\left(\left|\partial_x u\right|^2 + k\left|\partial_y u\right|^2\right)\right)\left(\xi I_1 \varrho_2 u^{\beta}\right) + \int_{\mathcal{R}} \left(\xi\left[\mathcal{L}, I_1 \varrho_2\right] u^{\beta}\right)\left(\xi I_1 \varrho_2 u^{\beta}\right) \\
&\quad = \mathcal{I} + \mathcal{J} + \mathcal{K}.
\end{aligned}$$

Now the term $|\mathcal{I}|^{\frac{1}{2}}$ is the first term in the conclusion of the lemma. For the second term, we write $I_1 = \partial \Lambda^{-2} = \left(\partial \Lambda^{-\frac{3}{2}}\right)\left(\Lambda^{-\frac{1}{2}}\right) = I_{\frac{1}{2}} I_{\frac{1}{2}}$ where ∂ is either ∂_x or ∂_y (we continue to abuse notation by writing $I_{\frac{1}{2}}$ for the three different operators $\partial_x \Lambda^{-\frac{3}{2}}$, $\partial_y \Lambda^{-\frac{3}{2}}$ and $\Lambda^{-\frac{1}{2}}$, each of order $-\frac{1}{2}$). We then obtain

$$\begin{aligned}
|\mathcal{J}| &= \beta(\beta-1) \int_{\mathbb{R}^2} \left(I_{\frac{1}{2}} \xi^2 I_1 \varrho_2 u^{\beta-2}\left(\left|\partial_x u\right|^2 + k\left|\partial_y u\right|^2\right)\right)\left(I_{\frac{1}{2}} \varrho_2 u^{\beta}\right) \\
&\leq C\beta(\beta-1) \int_{\mathbb{R}^2} \left|I_{\frac{1}{2}} \xi^2 I_1 \varrho_2 u^{\beta-2}\left(\left|\partial_x u\right|^2 + k\left|\partial_y u\right|^2\right)\right|^2 + C\beta(\beta-1) \int_{\mathbb{R}^2} \left|I_{\frac{1}{2}} \varrho_2 u^{\beta}\right|^2 \\
&\leq C\beta(\beta-1)\left\{\int_{\mathcal{R}} \left|\varrho_2 u^{\beta-2}\left(\left|\partial_x u\right|^2 + k\left|\partial_y u\right|^2\right)\right|\right\}^2 + C\beta(\beta-1)\left\{\int_{\mathcal{R}} \left|\varrho_2 u^{\beta}\right|^{\frac{4}{3}}\right\}^{\frac{3}{2}},
\end{aligned}$$

by Proposition 2.9 with first $T = I_{\frac{1}{2}} \xi^2 I_1$, $\alpha = -\frac{3}{2}$, $p = 1$ and $q = 2$, and then with $T = I_{\frac{1}{2}}$, $\alpha = -\frac{1}{2}$, $p = \frac{4}{3}$ and $q = 2$. Using Lemma 2.8 with $\frac{\beta}{2}$ in place of β in the case $\beta > 1$ ($\mathcal{J} = 0$ when $\beta = 1$), the first integral above satisfies

$$\begin{aligned}
&\int_{\mathcal{R}} \left|\varrho_2 u^{\beta-2}\left(\left|\partial_x u\right|^2 + k\left|\partial_y u\right|^2\right)\right| = C\frac{1}{\beta^2} \int_{\mathcal{R}} \left(\left|\sqrt{\varrho_2}\partial_x u^{\frac{\beta}{2}}\right|^2 + k\left|\sqrt{\varrho_2}\partial_y u^{\frac{\beta}{2}}\right|^2\right) \\
&\quad \leq C\frac{1}{\beta-1}\left|\int_{\mathcal{R}} \left(\sqrt{\varrho_2} u^{\frac{\beta}{2}-1} \mathcal{L} u\right)\left(\sqrt{\varrho_2} u^{\frac{\beta}{2}}\right)\right| + CA^2 \frac{1}{(\beta-1)^2} \int_{\mathcal{R}} \left|\xi u^{\beta}\right|.
\end{aligned}$$

So altogether, we have

$$|\mathcal{J}| \leq C\frac{\beta}{\beta-1}\left|\int_{\mathcal{R}} \left(\sqrt{\varrho_2} u^{\frac{\beta}{2}-1} \mathcal{L} u\right)\left(\sqrt{\varrho_2} u^{\frac{\beta}{2}}\right)\right|^2 + C\left[\frac{A^4 \beta}{(\beta-1)^3} + \beta(\beta-1)\right]\left(\int_{\mathcal{R}} \left|\xi u^{\beta}\right|^{\frac{4}{3}}\right)^{\frac{3}{2}},$$

where the first term here leads to (by taking the square root) the second term in the conclusion of the lemma.

Finally, to estimate the third term $\mathcal{K}$, we write $I_1 = \partial \Lambda^{-2} = \left(\partial \Lambda^{\alpha-2}\right)\left(\Lambda^{-\alpha}\right) = I_{1-\alpha} I_{\alpha}$ for any $0 < \alpha < 1$ to obtain

$$|\mathcal{K}| = \left|\int_{\mathcal{R}} \left(I_{1-\alpha} \xi^2\left[\mathcal{L}, I_1 \varrho_2\right] u^{\beta}\right)\left(I_{\alpha} \varrho_2 u^{\beta}\right)\right| \leq C \int_{\mathcal{R}} \left|I_{1-\alpha} \xi^2\left[\mathcal{L}, I_1 \varrho_2\right] u^{\beta}\right|^2 + C \int_{\mathcal{R}} \left|I_{\alpha} \varrho_2 u^{\beta}\right|^2.$$

As before,

$$\int_{\mathcal{R}} \left|I_{\alpha} \varrho_2 u^{\beta}\right|^2 \leq C_p\left(\int_{\mathcal{R}} \left|\varrho_2 u^{\beta}\right|^p\right)^{\frac{2}{p}}$$

for $\frac{1}{2} = \frac{1}{p} - \frac{\alpha}{2}$. We now write $\left[\mathcal{L}, I_1 \varrho_2\right] = \left[\mathcal{L}, I_1\right] \varrho_2 + I_1\left[\mathcal{L}, \varrho_2\right]$ and consider the two terms

$$\int_{\mathcal{R}} \left|I_{1-\alpha} \xi^2\left[\mathcal{L}, I_1\right] \varrho_2 u^{\beta}\right|^2 \text{ and } \int_{\mathcal{R}} \left|I_{1-\alpha} \xi^2 I_1\left[\mathcal{L}, \varrho_2\right] u^{\beta}\right|^2 \tag{2.14}$$

separately.

To estimate the first term in (2.14), we note that

$$[\mathcal{L}, I_1] = \partial_y [k, I_1] \partial_y = \partial_y (k I_1 \partial_y - I_1 k \partial_y) = \partial_y (k (I_1 \partial_y) - (I_1 \partial_y) k + I_1 k_y) = \partial_y ([k, I_1 \partial_y] + I_1 k_y).$$

Following [11], we denote by O_I^m the collection of rough pseudodifferential operators mapping $H_{compact}^{s+m,p}$ to $H_{loc}^{s,p}$ for $1 < p < \infty$ and $s \in I$, where $H^{s,p}$ denotes the Sobolev space of functions whose fractional derivatives up to order s lie in L^p. Now for $0 < \mu < 1$ and $\varepsilon > 0$ we have $\xi [k, I_1 \partial_y] \in O_{(-\varepsilon,\varepsilon)}^{-\mu}$ for $\mu + \varepsilon < \nu$ with norm $\|\varkappa k\|_{C^\nu(\mathbb{R}^2)}$ by Theorem 4 in [11]. Since $I_{1-\alpha} \xi \partial_y$ has order α, and since $\xi^2 \partial_y = \xi \partial_y \xi - \xi_y \xi$, we thus have

$$I_{1-\alpha} \xi^2 \partial_y [k, I_1 \partial_y] \in O_{(-\varepsilon+\alpha,\varepsilon-\alpha)}^{\alpha-\mu}, \qquad for\ 0 < \alpha < \min\{\mu, \varepsilon\},\ \mu + \varepsilon < \nu.$$

Thus $I_{1-\alpha} \xi^2 \partial_y [k, I_1 \partial_y]$ maps $L_{compact}^{p_1} = H_{compact}^{0,p_1}$ to $H_{loc}^{\mu-\alpha,p_1}$ provided $\mu - \alpha \in (-\varepsilon + \alpha, \varepsilon - \alpha)$, i.e. $\mu \in (2\alpha - \varepsilon, \varepsilon)$, which is in turn embedded in L_{loc}^2 by the Sobolev embedding theorem with $\frac{1}{2} = \frac{1}{p_1} - \frac{\mu-\alpha}{2}$. Note that given $\nu > 0$, we can first choose ε and α such that $0 < \frac{\varepsilon}{2} < \alpha < \varepsilon < \frac{\nu}{2}$, and then choose μ such that $\alpha < \mu < \varepsilon$, in order that all of the above parameter restrictions hold. So,

$$\begin{aligned}
\int_{\mathcal{R}} \left| I_{1-\alpha} \xi^2 [\mathcal{L}, I_1] \varrho_2 u^\beta \right|^2 &\leq C \int_{\mathcal{R}} \left| I_{1-\alpha} \xi^2 \partial_y [k, I_1 \partial_y] \varrho_2 u^\beta \right|^2 + C \int_{\mathcal{R}} \left| I_{1-\alpha} \xi^2 \partial_y I_1 k_2 \varrho_2 u^\beta \right|^2 \\
&\leq C \|\varkappa k\|_{C^\nu}^2 \left(\int_{\mathcal{R}} \left| \varrho_2 u^\beta \right|^{p_1} \right)^{\frac{2}{p_1}} + C \|\varkappa k_y\|_\infty^2 \left(\int_{\mathcal{R}} \left| \varrho_2 u^\beta \right|^{p_2} \right)^{\frac{2}{p_2}},
\end{aligned}$$

for $\frac{1}{2} = \frac{1}{p_1} - \frac{\mu-\alpha}{2}$ and $\frac{1}{2} = \frac{1}{p_2} - \frac{1-\alpha}{2}$ by Proposition 2.9.

To estimate the second term in (2.14), we observe that if T is defined by $T = I_1 ([\mathcal{L}, \varrho_2])$, then by expanding out $[\mathcal{L}, \varrho_2]$,

$$T = 2 \left(I_1 (\varrho_2)_x \partial_x + I_1 k (\varrho_2)_y \partial_y \right) + I_1 \left((\varrho_2)_{xx} + k (\varrho_2)_{yy} + k_y (\varrho_2)_y \right),$$

and then T is a bounded operator on L_{loc}^p with norm at most $CA (A + \|\varkappa k_2\|_\infty)$for all $1 < p < \infty$, and satisfies $T = T\xi$. Thus we have

$$\begin{aligned}
\int_{\mathcal{R}} \left| I_{1-\alpha} \xi^2 I_1 [\mathcal{L}, \varrho_2] u^\beta \right|^2 &= \int_{\mathcal{R}} \left| I_{1-\alpha} \xi^2 T u^\beta \right|^2 \leq C \left(\int_{\mathcal{R}} \left| T \xi u^\beta \right|^{p_2} \right)^{\frac{2}{p_2}} \\
&\leq CA^2 \left(A^2 + \|\varkappa k_2\|_\infty^2 \right) \left(\int_{\mathcal{R}} \left| \xi u^\beta \right|^{p_2} \right)^{\frac{2}{p_2}},
\end{aligned}$$

where $\frac{1}{2} = \frac{1}{p_2} - \frac{1-\alpha}{2}$ as above. This completes the proof of Lemma 2.10 if we take $p = \max\left\{p_1, p_2, \frac{4}{3}\right\}$.

3. A Nonlinear Degenerate Elliptic Equation

In this section we begin the proof of the *a priori* estimates (1.7) for smooth solutions of the quasilinear equation (1.1), which we recall here as

$$\|\zeta D^\alpha w\|_\infty \leq \mathcal{C}_\alpha (\|\varkappa \nabla w\|_\infty, L), \tag{3.1}$$

where $\mathcal{C}_\alpha (\cdot, \cdot)$ is finite and increasing on $[0, \infty) \times \mathcal{P}_c (\Omega)$, and w is smooth and satisfies

$$\mathcal{L} w = \left[\partial_x^2 + \partial_y k (x, w (x, y)) \partial_y \right] w = 0, \qquad (x, y) \in \Omega', \tag{3.2}$$

and also

$$(x, w(x,y)) \in L \text{ } for \text{ } all \text{ } (x,y) \in support(\varkappa). \tag{3.3}$$

Throughout this section, w will be a smooth solution of (3.2) satisfying (3.3), and for convenience, we will say that an expression involving derivatives of w is *under control* if it is dominated by the right side of (3.1). Similarly we will make statements to the effect that some derivative $D^\alpha w$ is in a Banach space $\mathcal{X}$ *with control*, meaning that $\|\zeta D^\alpha w\|_{\mathcal{X}}$ is *under control* for an appropriate cutoff function ζ.

We attack the problem by differentiating (3.2), to obtain the equations

$$\begin{aligned} 0 &= \mathcal{L}w_x + \partial_y \left[\{k_1(x, w(x,y)) + k_2(x, w(x,y)) w_x\} w_y \right], \\ 0 &= \mathcal{L}w_y + \partial_y \left[k_2(x, w(x,y)) w_y^2 \right], \end{aligned}$$

or

$$\begin{aligned} \mathcal{L}w_x &= -\partial_y k_1 w_y - \partial_y k_2 w_x w_y, \\ \mathcal{L}w_y &= -\partial_y k_2 w_y^2, \end{aligned} \tag{3.4}$$

for w_x and w_y. Note that we use ∂_y as an operator acting on everything to its right, unless parentheses indicate otherwise. Recall also Convention 1.2 concerning the expressions k, k_i etc. in this and subsequent sections: k denotes $k(x, w(x,y))$ and k_i denotes $k_i(x, w(x,y))$ etc., except in section 5 where k has more variables and the convention is modified accordingly.

We will apply Corollary 2.7 in the section on gradient estimates to the components of ∇w, and using the facts that both w and ∇w are bounded *with control*, we will show that in fact $w \in H^2$, i.e., $\nabla^2 w \in L^2$ *with control*. Note that this does not increase the index of smoothness of w that is *under control*, but only reverses the Sobolev embedding theorem $H^2(\mathbb{R}^2)$ " $\subset$ " $Lip_1(\mathbb{R}^2)$. Recall that the index of smoothness of an n-dimensional L^p Sobolev space $H_p^s(\mathbb{R}^n)$ is the quantity $s - \frac{n}{p}$. Since the equations (3.4) are not homogeneous, we must handle with care the terms arising from $\mathcal{L}\nabla w$ in applying Corollary 2.7. We then apply the results of the section on Moser iteration to obtain that $\nabla^2 w \in L^q$ *with control* for q large depending on how small R_1 is chosen, again handling with care the terms arising from $\mathcal{L}\nabla w$. Note that the Moser iteration actually *increases* the index of smoothness by $2\left(\frac{1}{2} - \frac{1}{q}\right) = 1 - \frac{2}{q}$.

At this point we repeat the above process with $\nabla^2 w$ in place of ∇w. We apply Corollary 2.7 in the section on gradient estimates to the components of $\nabla^2 w$, and using the facts that ∇w is bounded and $\nabla^2 w \in L^q$ *with control*, we show that in fact $w \in H^3$, i.e., $\nabla^3 w \in L^2$ *with control*. This time we actually increase the index of smoothness another $\frac{2}{q}$, for a total of 1. From now on, it turns out that due to the nature of the quasilinear systems satisfied by higher order gradients of w, which become progressively less nonlinear, we can continue to alternately apply the reverse Sobolev embedding and the Moser iteration to increase the index of smoothness of w that is *under control* by 1 with each repetition. Thus we obtain the *a priori* estimates (3.1).

3.1. **Reverse Sobolev embedding.** Here we show that if $\nabla w \in L^\infty$ *with control*, and satisfies the system (3.4), then $\nabla^2 w \in L^2$ *with control*. The following lemmas will be crucial in handling the nonhomogeneous terms in (3.4).

Lemma 3.1. *Suppose w is a smooth solution of (3.2) in a compact rectangle $\mathcal{R}$ in Ω', where $k(x,y)$ is smooth and nonnegative in Ω, so that $u = w_x$ and $v = w_y$ are smooth solutions in $\mathcal{R}$ of the nonlinear system (3.4). Then we have*

$$\begin{aligned} &\int_{\mathcal{R}} \left(|\partial_x \zeta u|^2 + k |\partial_y \zeta u|^2 \right) dxdy + \int_{\mathcal{R}} \left(|\partial_x \zeta v|^2 + k |\partial_y \zeta v|^2 \right) dxdy \\ \leq \quad & CA^2 \left(\|\xi u\|_{L^2}^2 + \|\xi v\|_{L^2}^2 \right) + CB^2 \left(\|\xi u\|_{L^4}^4 + \|\xi v\|_{L^4}^4 \right). \end{aligned}$$

Alternatively, we have a bound in terms of at most $\|\xi u\|_{L^2}$ and $\|\xi v\|_{L^\infty}$;

$$\begin{aligned} &\int_{\mathcal{R}} \left(|\partial_x \zeta u|^2 + k |\partial_y \zeta u|^2 \right) dxdy + \int_{\mathcal{R}} \left(|\partial_x \zeta v|^2 + k |\partial_y \zeta v|^2 \right) dxdy \\ \leq \quad & CA^2 \left(\|\xi u\|_{L^2}^2 + \|\xi v\|_{L^2}^2 \right) + CB^2 \left(\|\xi u\|_{L^2}^2 \|\xi v\|_{L^\infty}^2 + \|\xi v\|_{L^4}^4 \right). \end{aligned}$$

Proof. From Corollary 2.2, applied with $k(x, w(x,y))$ in place of $k(x,y)$ there, we have

$$\begin{aligned} &\int_{\mathcal{R}} \left(|\partial_x \zeta u|^2 + k |\partial_y \zeta u|^2 \right) + \int_{\mathcal{R}} \left(|\partial_x \zeta v|^2 + k |\partial_y \zeta v|^2 \right) \\ &\qquad \leq -4 \int_{\mathcal{R}} (\zeta \mathcal{L} u)(\zeta u) - 4 \int_{\mathcal{R}} (\zeta \mathcal{L} v)(\zeta v) + CA^2 \|\xi u\|_0^2 + CA^2 \|\xi v\|_0^2 . \end{aligned} \tag{3.5}$$

For the integral involving $\mathcal{L}v$, we have by (3.4)

$$\begin{aligned} -\int_{\mathcal{R}} (\zeta \mathcal{L} v)(\zeta v) &= \int_{\mathcal{R}} \left(\zeta \partial_y k_2 v^2 \right)(\zeta v) = -\int_{\mathcal{R}} \left(v^2 \right) \left(k_2 \partial_y \zeta^2 v \right) \\ &= -\int_{\mathcal{R}} \left(v^2 \right) \left(k_2 \zeta \partial_y \zeta v \right) - \int_{\mathcal{R}} \left(v^2 \right) \left(k_2 \zeta_y \zeta v \right) \\ &= -\int_{\mathcal{R}} \left(\zeta v^2 \right) \left(k_2 \partial_y \zeta v \right) - \int_{\mathcal{R}} \left(\zeta v^2 \right) \left(k_2 \zeta_y v \right). \end{aligned}$$

The first term on the right is dominated by

$$B^2 C \varepsilon \int_{\mathcal{R}} k |\partial_y \zeta v|^2 + \frac{C}{\varepsilon} \|\xi v\|_{L^4}^4 ,$$

while the second term is at most $CB^2 \|\xi v\|_{L^4}^4 + CA^2 \|\xi v\|_{L^2}^2$. Choosing $\varepsilon = \frac{1}{2CB^2}$, we can absorb the term $B^2 C \varepsilon \int_{\mathcal{R}} k |\partial_y \zeta v|^2$ into the left side of (3.5). The same argument yields the appropriate estimate for

$$-\int_{\mathcal{R}} (\zeta \mathcal{L} u)(\zeta u) = \int_{\mathcal{R}} (\zeta \partial_y k_1 v)(\zeta u) + \int_{\mathcal{R}} (\zeta \partial_y k_2 u v)(\zeta u).$$

To obtain the alternate bound, we estimate the last integral above by

$$\begin{aligned} \left| \int_{\mathcal{R}} (\zeta \partial_y k_2 u v)(\zeta u) \right| &= \left| -\int_{\mathcal{R}} (\zeta u v)(k_2 \partial_y \zeta u) - \int_{\mathcal{R}} (k_2 \zeta u v) \left(\zeta_y u \right) \right| \\ &\leq \frac{C}{\varepsilon} \|\xi v\|_{L^\infty}^2 \|\zeta u\|_{L^2}^2 + B^2 C \varepsilon \int_{\mathcal{R}} k |\partial_y \zeta u|^2 \\ &\quad + CA^2 \|\xi u\|_{L^2}^2 + CB^2 \|\xi u\|_{L^2}^2 \|\xi v\|_{L^\infty}^2 , \end{aligned}$$

and similarly for the other integral. Choosing $\varepsilon = \frac{1}{2CB^2}$ again completes the proof of the lemma.

Up to this point we have been keeping precise track of all the constants. This will prove increasingly difficult from now on, and we will instead only keep close track of the critical constants - typically those which are involved in subsequent absorptions.

Lemma 3.2. *Suppose w is a smooth solution of (3.2) in $\mathcal{R}$, where $k(x,y)$ is non-negative and smooth in Ω and satisfies (1.5) so that $u = w_x$ and $v = w_y$ are smooth solutions in $\mathcal{R}$ of the nonlinear system (3.4). Then we have with $\partial = \partial_x$ or $\partial = \partial_y$,*

$$\int_{\mathcal{R}} \left(|\partial_x (\zeta \partial \eta u)|^2 + k |\partial_y (\zeta \partial \eta u)|^2 \right) dxdy + \int_{\mathcal{R}} \left(|\partial_x (\zeta \partial \eta v)|^2 + k |\partial_y (\zeta \partial \eta v)|^2 \right) dxdy$$
$$\leq \mathcal{C}(B, \|\varkappa \nabla w\|_\infty) \left(\|\eta u\|_1^2 + \|\eta v\|_1^2 \right) + \mathcal{C}(A, B, \|\varkappa \nabla w\|_\infty),$$

where the functions $\mathcal{C}(\cdot,\cdot)$ and $\mathcal{C}(\cdot,\cdot,\cdot)$ are finite and increasing in each variable separately.

Proof. We wish to apply Corollary 2.7 with $k(x,y)$ replaced by $k = k(x, w(x,y))$. Now by (1.5), we have

$$|\nabla k| = |(k_1 + k_2 w_x, k_2 w_y)| \leq |k_1| + |k_2 \nabla w| \leq CB\sqrt{k}\,(1 + |\nabla w|), \tag{3.6}$$

and thus we can apply Corollary 2.7 if we replace B by $\widetilde{B} = CB(1 + \|\varkappa \nabla w\|_\infty)$. We obtain

$$\begin{aligned} &\int_{\mathcal{R}} \left(|\partial_x (\zeta \partial \eta u)|^2 + k |\partial_y (\zeta \partial \eta u)|^2 \right) + \int_{\mathcal{R}} \left(|\partial_x (\zeta \partial \eta v)|^2 + k |\partial_y (\zeta \partial \eta v)|^2 \right) \\ &\leq -4 \int_{\mathcal{R}} (\zeta \partial \eta \mathcal{L} u)(\zeta \partial \eta u) - 4 \int_{\mathcal{R}} (\zeta \partial \eta \mathcal{L} v)(\zeta \partial \eta v) \\ &+ CA^2 \left| \int_{\mathcal{R}} (\eta \mathcal{L} u)(\eta u) \right| + CA^2 \left| \int_{\mathcal{R}} (\eta \mathcal{L} v)(\eta v) \right| \\ &+ CA^4 \left\{ \|\varrho_1 u\|_1^2 + \|\varrho_1 v\|_1^2 \right\} \\ &+ CA^2 \left(A^2 + \tilde{B}^2 \right) \left\{ \int_{\mathcal{R}} k |\partial_y \varrho_2 u|^2 + \int_{\mathcal{R}} k |\partial_y \varrho_2 v|^2 \right\} \\ &+ CA^2 \left\{ \int_{\mathcal{R}} k |\partial \varrho_2 u|^2 + \int_{\mathcal{R}} k |\partial \varrho_2 v|^2 \right\} + C\left(1 + \tilde{B}^2\right) \left\{ \|\eta u\|_1^2 + \|\eta v\|_1^2 \right\} \\ &+ CA^4 \left(A^4 + \tilde{B}^2 \right) \left\{ \|\xi u\|_0^2 + \|\xi v\|_0^2 \right\}, \end{aligned} \tag{3.7}$$

for $\partial = \partial_x$ or ∂_y. We first estimate

$$\begin{aligned} -\int_{\mathcal{R}} (\zeta \partial \eta \mathcal{L} v)(\zeta \partial \eta v) &= \int_{\mathcal{R}} \left(\zeta \partial \eta \partial_y k_2 v^2 \right)(\zeta \partial \eta v) = \int_{\mathcal{R}} \left(v^2 \right) \left(k_2 \partial_y \eta \partial \zeta^2 \partial \eta v \right) \\ &= \int_{\mathcal{R}} \left(v^2 \right) \left(\eta \partial \zeta^2 k_2 \partial_y \partial \eta v \right) + \int_{\mathcal{R}} \left(v^2 \right) \left(\left[k_2 \partial_y, \eta \partial \zeta^2 \right] \partial \eta v \right) \\ &= -\int_{\mathcal{R}} \left(\zeta \partial \eta v^2 \right) \left(k_2 \partial_y \zeta \partial \eta v \right) + \int_{\mathcal{R}} \left(v^2 \right) \left(\left[k_2 \partial_y, \eta \partial \zeta^2 \right] \partial \eta v \right) \\ &\quad + \int_{\mathcal{R}} \left(\zeta \partial \eta v^2 \right) \left(k_2 \zeta_y \partial \eta v \right) \\ &= I + II + III. \end{aligned}$$

For term I we use

$$|I| \leq \frac{C}{\varepsilon} \int_{\mathcal{R}} \left| \zeta \partial \eta v^2 \right|^2 + \tilde{B}^2 \varepsilon \int_{\mathcal{R}} k \left| \partial_y \zeta \partial \eta v \right|^2,$$

since $|k_2| \leq \tilde{B}\sqrt{k}$ by (1.5), and absorb the second term $\tilde{B}^2\varepsilon \int_{\mathcal{R}} k \left|\partial_y \zeta \partial \eta v\right|^2$ into the left side of (3.7) upon choosing $\varepsilon = \frac{1}{2\tilde{B}^2}$. As for the first term, since $\partial \eta v^2 = 2v\partial\eta v - (\partial \eta) v^2$, we have

$$\begin{aligned}\int_{\mathcal{R}} \left|\partial \eta v^2\right|^2 &\leq C\int_{\mathcal{R}} \left|v\partial\eta v\right|^2 + C\int_{\mathcal{R}} \left|(\partial\eta) v^2\right|^2 \\ &\leq C\left\|\xi v\right\|_{L^\infty}^2 \left\|\eta v\right\|_1^2 + CA^2 \left\|\xi v\right\|_{L^4}^4 .\end{aligned}$$

Now use $\|\xi v\|_{L^4}^4 \leq |\mathcal{R}| \, \|\xi v\|_{L^\infty}^4 \leq C \|\xi v\|_{L^\infty}^4$ and multiply the resulting terms above by $\frac{C}{\varepsilon} = 2C\widetilde{B}^2$ to obtain an expression which is bounded by the right side of the conclusion of Lemma 3.2.

For term III, we use

$$|III| = \left|\int_{\mathcal{R}} \left(\zeta\partial\eta v^2\right)\left(k_2\zeta_y\partial\eta v\right)\right| \leq C\int_{\mathcal{R}} \left|\partial\eta v^2\right|^2 + CA^2\widetilde{B}^2\int_{\mathcal{R}} k\left|\partial\eta v\right|^2 .$$

The first term is handled by the previous inequality, and the second is at most

$$C\left(A^2\tilde{B}^2\right)\left(A^2 + \tilde{B}^2 \left\|\xi v\right\|_{L^\infty}^2\right)\left\|\xi v\right\|_{L^\infty}^2$$

by Lemma 3.1.

For term II,

$$\left[k_2\partial_y, \eta\partial\zeta^2\right] = k_2\eta_y\partial\zeta^2 + k_2\eta\partial 2\zeta\zeta_y - \eta\zeta^2\left(\partial k_2\right)\partial_y,$$

implies

(3.8)

$$|II| \leq \left|\int_{\mathcal{R}} \left(v^2\right)\left(k_2\eta_y\partial\zeta^2\partial\eta v\right)\right| + \left|\int_{\mathcal{R}} \left(v^2\right)\left(k_2\eta\partial 2\zeta\zeta_y\partial\eta v\right)\right| + \left|\int_{\mathcal{R}} \left(v^2\right)\left(\eta\zeta^2\left(\partial k_2\right)\partial_y\partial\eta v\right)\right| .$$

Now the first of the terms in (3.8) satisfies

$$\begin{aligned}\left|\int_{\mathcal{R}} \left(v^2\right)\left(k_2\eta_y\partial\zeta^2\partial\eta v\right)\right| &\leq \left|\int_{\mathcal{R}} \left(v^2\right)\left(k_2\eta_y\zeta\partial\zeta\partial\eta v\right)\right| + \left|\int_{\mathcal{R}} \left(v^2\right)\left(k_2\eta_y\left(\partial\zeta\right)\zeta\partial\eta v\right)\right| \\ &\leq \tilde{B}^2 C\varepsilon\int_{\mathcal{R}} k\left|\partial\zeta\partial\eta v\right|^2 + \frac{C}{\varepsilon}\int_{\mathcal{R}} \left|\eta_y v^2\right|^2 \\ &\quad + \tilde{B}^2 C\int_{\mathcal{R}} k\left|\eta_y\left(\partial\zeta\right)v^2\right|^2 + C\int_{\mathcal{R}} \left|\partial\eta v\right|^2 \\ &\leq \tilde{B}^2 C\varepsilon\int_{\mathcal{R}} k\left|\partial\zeta\partial\eta v\right|^2 + \frac{C}{\varepsilon}A^2\left\|\xi v\right\|_{L^4}^4 + CA^4\tilde{B}^2\left\|\xi v\right\|_{L^4}^4 + C\left\|\eta v\right\|_1^2 ,\end{aligned}$$

and the first term on the right above can be absorbed into the left side of (3.7) with $\varepsilon = \frac{1}{2C\tilde{B}^2}$. The second term in (3.8) can be handled in exactly the same way. The third term in (3.8) is handled as follows:

$$\begin{aligned}&\left|\int_{\mathcal{R}} \left(v^2\right)\left(\eta\zeta^2\left(\partial k_2\right)\partial_y\partial\eta v\right)\right| \\ &\leq \left|\int_{\mathcal{R}} \left(v^2\right)\left(\eta\zeta^2\left(k_{21} + k_{22}u\right)\partial_y\partial_x\eta v\right)\right| + \left|\int_{\mathcal{R}} \left(v^2\right)\left(\eta\zeta^2\left(k_{22}v\right)\partial_y\partial_y\eta v\right)\right| \\ &\leq C\int_{\mathcal{R}} \left|\partial_y k_{21}\zeta^2\eta v^2\right|^2 + C\int_{\mathcal{R}} \left|\partial_y k_{22}u\zeta^2\eta v^2\right|^2 + C\int_{\mathcal{R}} \left|\partial_y k_{22}v\zeta^2\eta v^2\right|^2 + C\left\|\eta v\right\|_1^2 .\end{aligned}$$

The first three integrals on the right are now easily dominated by

$$C\left\|\xi\nabla w\right\|_{L^\infty}^4\left(\left\|\xi\nabla w\right\|_{L^\infty}^4+A^2\left\|\xi\nabla w\right\|_{L^\infty}^2+\left\|\eta u\right\|_1^2+\left\|\eta v\right\|_1^2\right)$$

using

$$\partial\zeta^2 k_{22}\eta uv^2=\left(\partial\zeta^2 k_{22}\right)\eta uv^2+\zeta^2 k_{22}\left[v^2\partial\eta u+u2v\partial\eta v-2uv^2\partial\eta\right]$$

and

$$\begin{aligned}\partial\zeta^2 k_{22} &= 2\zeta\left(\partial\zeta\right)k_{22}+\zeta^2\partial k_{22}\\ &= \begin{cases}2\zeta\zeta_x k_{22}+\zeta^2\left(k_{221}+k_{222}u\right), & \partial=\partial_x\\ 2\zeta\zeta_y k_{22}+\zeta^2 k_{222}v, & \partial=\partial_y\end{cases},\end{aligned}$$

along with similar formulas for the terms involving $k_{21}v^2$ and $k_{22}v^3$. This completes the estimates for the second term on the right side of (3.7). Similar arguments handle the first term on the right side of (3.7).

Next, we turn to estimating $\left|\int_{\mathcal{R}}\left(\eta\mathcal{L}u\right)\left(\eta u\right)\right|+\left|\int_{\mathcal{R}}\left(\eta\mathcal{L}v\right)\left(\eta v\right)\right|$. For this we have

$$\begin{aligned}\left|\int_{\mathcal{R}}\left(\eta\mathcal{L}v\right)\left(\eta v\right)\right| &= \left|\int_{\mathcal{R}}\left(\eta\partial_y k_2 v^2\right)\left(\eta v\right)\right|=\left|-\int_{\mathcal{R}}\left(\eta v^2\right)\left(k_2\partial_y\eta v\right)-\int_{\mathcal{R}}\left(k_2\eta\eta_y v^3\right)\right|\\ &\leq \tilde{B}^2\int_{\mathcal{R}}k\left|\partial_y\eta v\right|^2+C\int_{\mathcal{R}}\left|\eta v^2\right|^2+CA\left|\mathcal{R}\right|\left\|\zeta v\right\|_\infty^3,\end{aligned}$$

and the first term on the right is controlled by Lemma 3.1. A similar argument applies to $\left|\int_{\mathcal{R}}\left(\eta\mathcal{L}u\right)\left(\eta u\right)\right|$.

Finally, we turn to the remaining terms in (3.7) that arose from the application of Corollary 2.7. The terms

$$\int_{\mathcal{R}}k\left|\partial_y\varrho_2 u\right|^2+\int_{\mathcal{R}}k\left|\partial_y\varrho_2 v\right|^2 \text{ and } \int_{\mathcal{R}}k\left|\partial\varrho_2 u\right|^2+\int_{\mathcal{R}}k\left|\partial\varrho_2 v\right|^2$$

are handled by Lemma 3.1, while the terms $\left\|\varrho_1 u\right\|_1^2+\left\|\varrho_1 v\right\|_1^2$ are handled by elliptic theory (since ϱ_1 is supported where $k>0$) as given in the Proposition below. The penultimate term in (3.7) is included in the first term on the right side of the conclusion, while the final term in (3.7) is included in the second term. This completes the proof of the lemma.

Proposition 3.3. *Suppose $k\geq c>0$ is smooth and ζ,ξ are smooth cutoff functions with $\xi=1$ on the support of ζ. For each multiindex α, there is a finite increasing function $\mathcal{C}_\alpha(\cdot)$ on $[0,\infty)$, such that*

$$\left\|\zeta D^\alpha w\right\|_{L^\infty}\leq\mathcal{C}_\alpha\left(\left\|\xi w\right\|_{L^\infty}+\left\|\xi\nabla w\right\|_{L^\infty}\right),$$

for all smooth solutions w of

$$\partial_x^2 w+\partial_y k\left(x,w\right)\partial_y w=0. \tag{3.9}$$

Proof. We write (3.9) in nondivergence form as follows:

$$\partial_x^2 w+k\left(x,w\right)\partial_y^2 w=-k_2\left(x,w\right)\left(\partial_y w\right)^2=f. \tag{3.10}$$

Then $k\left(x,w\right)$ and f are bounded functions with $k\left(x,w\right)\geq c>0$, and so by Theorem 12.4 in [5], we conclude that for some $\delta>0$,

$$\left\|\zeta w\right\|_{C^{1+\delta}}\leq C\left(\left\|\xi w\right\|_{L^\infty}+\left\|\xi\frac{f}{c}\right\|_{L^\infty}\right)\leq C_1\left(\left\|\xi w\right\|_{L^\infty}+\left\|\xi\nabla w\right\|_{L^\infty}^2\right).$$

Now return to (3.10) and note that $f \in C^{\delta}$ and $k(x,w) \in C^{1+\delta}$ *with control.* By the Schauder estimates, Theorem 6.2 in [5], we obtain

$$\|\zeta w\|_{C^{2+\delta}} \leq \mathcal{C}_2 \left(\|\xi w\|_{L^\infty}, \|\xi \nabla w\|_{L^\infty} \right),$$

and so also $k(x,w) \in C^{2+\delta}$ and $f \in C^{1+\delta}$ *with control.* We can now differentiate (3.10) with respect to ∂ and apply Schauder theory again to obtain

$$\|\zeta w\|_{C^{3+\delta}} \leq \mathcal{C}_3 \left(\|\xi w\|_{L^\infty}, \|\xi \nabla w\|_{L^\infty} \right).$$

Iterating this process yields the conclusion of the proposition.

Theorem 3.4. *Suppose w is a smooth solution of (1.1),*

$$\partial_x^2 w + \partial_y k(x, w(x,y)) \partial_y w = 0$$

in $\mathcal{R}$, where k is nonnegative, smooth and satisfies (1.5). Then $w \in H^2_{loc}$ with control, i.e. $w, \nabla^2 w \in L^2_{loc}$ with control.

Proof. The Poincaré inequality (2.3) and Lemma 3.2 yield with $u = \partial_x w$ and $v = \partial_y w$,

$$\begin{aligned} \|\eta u\|_1^2 + \|\eta v\|_1^2 &\leq C R_1^2 \int_{\mathcal{R}} \left(|\partial_x (\zeta \nabla \eta u)|^2 + |\partial_x (\zeta \nabla \eta v)|^2 \right) \\ &\leq C R_1^2 \left(1 + \left(B^2+1\right) \|\xi \nabla w\|_\infty^4 \right) \left(\|\eta u\|_1^2 + C \|\eta v\|_1^2 \right) + C R_1^2 \mathcal{C}\left(A, B, \|\xi \nabla w\|_\infty\right). \end{aligned}$$

Choosing $R_1 \leq \left\{ 2C \left(1 + \left(B^2+1\right) \|\xi \nabla w\|_\infty^4 \right) \right\}^{-\frac{1}{2}}$ (note that A is not involved here) permits the first term on the right above to be absorbed into the left hand side, and this completes the proof of the theorem.

3.2. **An L^p improvement.** In this subsection, we improve the index of smoothness of w that is *under control* by showing that $\nabla^2 w \in L^q$ *with control* for large $q > 2$. Let us first compute the equations satisfied by the L^2 functions $\nabla^2 w$. Differentiating (3.4), and continuing to set $u = w_x$ and $v = w_y$, yields

$$\begin{aligned} 0 &= \mathcal{L} u_x + \partial_y \left\{ (k_1 + k_2 u) u_y \right\} \\ &\quad + \partial_y \left\{ (k_{11} + k_{12} u) v + k_1 v_x + (k_{12} + k_{22} u) uv + k_2 u_x v + k_2 u v_x \right\} \\ 0 &= \mathcal{L} u_y + \partial_y \left\{ k_2 v u_y \right\} + \partial_y \left\{ (k_{12} v) v + k_1 v_y + (k_{22} v) uv + k_2 u_y v + k_2 u v_y \right\} \\ 0 &= \mathcal{L} v_x + \partial_y \left\{ (k_1 + k_2 u) v_y \right\} + \partial_y \left\{ (k_{12} + k_{22} u) v^2 + k_2 2 v v_x \right\} \\ 0 &= \mathcal{L} v_y + \partial_y \left\{ k_2 v v_y \right\} + \partial_y \left\{ (k_{22} v) v^2 + k_2 2 v v_y \right\} \end{aligned}$$

or

$$\begin{aligned} -\mathcal{L} u_x &= \partial_y \left\{ \begin{array}{c} k_1 (u_y + v_x) + k_2 (u u_y + u_x v + u v_x) \\ + k_{11} v + 2 k_{12} u v + k_{22} u^2 v \end{array} \right\} \qquad (3.11) \\ -\mathcal{L} u_y &= \partial_y \left\{ k_1 v_y + k_2 (u v_y + 2 u_y v) + k_{12} v^2 + k_{22} u v^2 \right\} \\ -\mathcal{L} v_x &= \partial_y \left\{ k_1 v_y + k_2 (u v_y + 2 v v_x) + k_{12} v^2 + k_{22} u v^2 \right\} \\ -\mathcal{L} v_y &= \partial_y \left\{ 3 k_2 v v_y + k_{22} v^3 \right\}. \end{aligned}$$

The key feature of this system is that the right hand side is a combination of terms involving either the operator $\partial_y k_i = (k_i \partial_y)^t$, the transpose of the subunit vector field $k_i \partial_y$, or the identity operator acting on an expression which is affine in the components of ∇u and ∇v with bounded coefficients. We rewrite this system so as to exploit this feature as follows:

$$\begin{aligned} (3.12)\quad -\mathcal{L}u_x &= (k_1\partial_y)^t (u_y+v_x) + (k_2\partial_y)^t (uu_y+u_xv+uv_x) \\ &\quad + \left\{ \begin{array}{c} k_{112}v^2 + 2k_{122}uv^2 + k_{222}u^2v^2 + k_{11}v_y \\ +2k_{12}(u_yv+uv_y) + k_{22}\left(2uu_yv+u^2v_y\right) \end{array} \right\} \\ -\mathcal{L}u_y &= (k_1\partial_y)^t v_y + (k_2\partial_y)^t (uv_y+2u_yv) \\ &\quad + \left\{ k_{122}v^3 + k_{222}uv^3 + k_{12}2vv_y + k_{22}\left(u_yv^2+2uvv_y\right)\right\} \\ -\mathcal{L}v_x &= (k_1\partial_y)^t v_y + (k_2\partial_y)^t (uv_y+2vv_x) \\ &\quad + \left\{ k_{122}v^3 + k_{222}uv^3 + k_{12}2vv_y + k_{22}\left(u_yv^2+2uvv_y\right)\right\} \\ -\mathcal{L}v_y &= (k_2\partial_y)^t 3vv_y + \left\{k_{222}v^4 + k_{22}3v^2v_y\right\}, \end{aligned}$$

where we recall that the derivatives of k are evaluated at $(x, w(x,y))$. The following lemma is crucial for estimating the nonlinear terms above. We recall that limiting arguments show that expressions like $\partial_y u_y^\beta = \beta u_y^{\beta-1}u_{yy}$ are square integrable for $\beta > \frac{1}{2}$ (and not just $\beta > 1$).

Lemma 3.5. *Suppose that u_x, u_y, v_x, v_y give a smooth solution of the system (3.11) in $\mathcal{R}$ with $k = k(x, w(x,y))$. Then for $\beta > \frac{1}{2}$, the k-gradient integrals*

$$\begin{aligned} (3.13)\quad & \int_{\mathcal{R}} \left(\left|\zeta\partial_x u_x^\beta\right|^2 + k\left|\zeta\partial_y u_x^\beta\right|^2 \right) + \int_{\mathcal{R}} \left(\left|\zeta\partial_x u_y^\beta\right|^2 + k\left|\zeta\partial_y u_y^\beta\right|^2 \right) \\ &+ \int_{\mathcal{R}} \left(\left|\zeta\partial_x v_x^\beta\right|^2 + k\left|\zeta\partial_y v_x^\beta\right|^2 \right) + \int_{\mathcal{R}} \left(\left|\zeta\partial_x v_y^\beta\right|^2 + k\left|\zeta\partial_y v_y^\beta\right|^2 \right) \end{aligned}$$

are dominated by

$$\begin{aligned} \leq\ & \mathcal{C}_1\left(\beta, \frac{1}{\beta-\frac{1}{2}}, B, \|\xi\nabla w\|_\infty\right) \int_{\mathcal{R}} \left\{ \left|\zeta u_x^\beta\right|^2 + \left|\zeta u_y^\beta\right|^2 + \left|\zeta v_x^\beta\right|^2 + \left|\zeta v_y^\beta\right|^2 \right\} \\ & + C\left(\frac{\beta}{2\beta-1}\right)^2 A^2 \int_{\mathcal{R}} \left\{ \left|\varrho_1 u_x^\beta\right|^2 + \left|\varrho_1 u_y^\beta\right|^2 + \left|\varrho_1 v_x^\beta\right|^2 + \left|\varrho_1 v_y^\beta\right|^2 \right\} \\ & + \mathcal{C}_2\left(\beta, \frac{1}{\beta-\frac{1}{2}}, A, B, \|\xi\nabla w\|_\infty\right) \int_{\mathcal{R}} k \left\{ \left|\varrho_2 u_x^\beta\right|^2 + \left|\varrho_2 u_y^\beta\right|^2 + \left|\varrho_2 v_x^\beta\right|^2 + \left|\varrho_2 v_y^\beta\right|^2 \right\} \\ & + \mathcal{C}_3\left(\beta, \frac{1}{\beta-\frac{1}{2}}, \|\xi\nabla w\|_\infty\right), \end{aligned}$$

where $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$ are finite and increasing in each variable separately.

A crucial point is that $\mathcal{C}_1$ in the above lemma does not depend on A, so that in applying the one dimensional Poincaré inequality in the next theorem, the product $R_1^2\mathcal{C}_1$ can be made less than one for R_1 sufficiently small. This would be impossible if A^2 were present since $A \geq R_1^{-1}$ - recall (2.7).

Proof. We see from Lemma 2.8 applied to the four functions u_x, u_y, v_x, v_y, that it suffices to prove that

$$\begin{aligned} (3.14)\quad & \left|\int_{\mathcal{R}} (\zeta\mathcal{L}u_x)\left(\zeta u_x^{2\beta-1}\right)\right| + \left|\int_{\mathcal{R}} (\zeta\mathcal{L}u_y)\left(\zeta u_y^{2\beta-1}\right)\right| \\ & + \left|\int_{\mathcal{R}} (\zeta\mathcal{L}v_x)\left(\zeta v_x^{2\beta-1}\right)\right| + \left|\int_{\mathcal{R}} (\zeta\mathcal{L}v_y)\left(\zeta v_y^{2\beta-1}\right)\right| \end{aligned}$$

is dominated by

$$\begin{aligned}
&C\alpha\tilde{B}^2\int_{\mathcal{R}}k\left\{\left|\zeta\partial_y u_x^\beta\right|^2+\left|\zeta\partial_y u_y^\beta\right|^2+\left|\zeta\partial_y v_x^\beta\right|^2+\left|\zeta\partial_y v_y^\beta\right|^2\right\}\\
&+\mathcal{C}\left(\beta,\frac{1}{\beta-\frac{1}{2}},B,\|\xi\nabla w\|_\infty,\frac{1}{\alpha}\right)\int_{\mathcal{R}}\left\{\left|\zeta u_x^\beta\right|^2+\left|\zeta u_y^\beta\right|^2+\left|\zeta v_x^\beta\right|^2+\left|\zeta v_y^\beta\right|^2\right\}\\
&+\mathcal{C}\left(\beta,\frac{1}{\beta-\frac{1}{2}},A,B,\|\xi\nabla w\|_\infty\right)\int_{\mathcal{R}}k\left\{\left|\varrho_2 u_x^\beta\right|^2+\left|\varrho_2 u_y^\beta\right|^2+\left|\varrho_2 v_x^\beta\right|^2+\left|\varrho_2 v_y^\beta\right|^2\right\}\\
&+\mathcal{C}\left(\|\xi\nabla w\|_\infty\right),
\end{aligned}$$

for any $0<\alpha<1$, and where the function $\mathcal{C}$ is finite and increasing in each variable separately. Indeed, then the terms

$$C\alpha\tilde{B}^2\frac{\beta^2}{2\beta-1}\int_{\mathcal{R}}k\left\{\left|\zeta\partial_y u_x^\beta\right|^2+\left|\zeta\partial_y u_y^\beta\right|^2+\left|\zeta\partial_y v_x^\beta\right|^2+\left|\zeta\partial_y v_y^\beta\right|^2\right\}$$

can be absorbed into the left side of (3.13) for $\alpha=\frac{2\beta-1}{2C\tilde{B}^2\beta^2}$. Let us illustrate the bound for the term $\left|\int_{\mathcal{R}}\left(\zeta\mathcal{L}v_y\right)\left(\zeta v_y^{2\beta-1}\right)\right|$, which is given by

$$\begin{aligned}
(3.15)\quad&\left|\int_{\mathcal{R}}\zeta\left(\left(k_2\partial_y\right)^t 3vv_y+\left\{k_{222}v^4+k_{22}3v^2v_y\right\}\right)\left(\zeta v_y^{2\beta-1}\right)\right|\\
&\le\ C\left|\int_{\mathcal{R}}\zeta\left(\left(k_2\partial_y\right)^t 3vv_y\right)\left(\zeta v_y^{2\beta-1}\right)\right|+C\left|\int_{\mathcal{R}}\zeta\left(k_{222}v^4+k_{22}3v^2v_y\right)\left(\zeta v_y^{2\beta-1}\right)\right|.
\end{aligned}$$

Now the first term here satisfies

$$\begin{aligned}
\left|\int_{\mathcal{R}}\zeta\left(\left(k_2\partial_y\right)^t 3vv_y\right)\left(\zeta v_y^{2\beta-1}\right)\right| &= \left|\int_{\mathcal{R}}\left(3vv_y\right)\left(k_2\partial_y\zeta^2v_y^{2\beta-1}\right)\right|\\
&\le C\left|\int_{\mathcal{R}}\left(3vv_y\right)\left(\zeta^2v_y^{\beta-1}k_2\partial_y v_y^\beta\right)\right|\\
&\quad+C\left|\int_{\mathcal{R}}\left(3vv_y\right)\left(\zeta v_y^{\beta-1}k_2\zeta_y v_y^\beta\right)\right|,
\end{aligned}$$

since $\frac{2\beta-1}{\beta}$ is bounded. Estimating these two terms separately, we have

$$\begin{aligned}
\left|\int_{\mathcal{R}}\left(3vv_y\right)\left(\zeta^2v_y^{\beta-1}k_2\partial_y v_y^\beta\right)\right| &= \left|\int_{\mathcal{R}}\left(3v\zeta v_y^\beta\right)\left(\zeta k_2\partial_y v_y^\beta\right)\right|\\
&\le \frac{C}{\alpha}\|\xi v\|_{L^\infty}^2\int_{\mathcal{R}}\left|\zeta v_y^\beta\right|^2+\alpha\tilde{B}^2\int_{\mathcal{R}}k\left|\zeta\partial_y v_y^\beta\right|^2,
\end{aligned}$$

and

$$\begin{aligned}
\left|\int_{\mathcal{R}}\left(3vv_y\right)\left(\zeta v_y^{\beta-1}k_2\zeta_y v_y^\beta\right)\right| &= \left|\int_{\mathcal{R}}\left(3v\zeta v_y^\beta\right)\left(\zeta_y k_2 v_y^\beta\right)\right|\\
&\le C\|\xi v\|_{L^\infty}^2\int_{\mathcal{R}}\left|\zeta v_y^\beta\right|^2+CA^2\tilde{B}^2\int_{\mathcal{R}}k\left|\varrho_2 v_y^\beta\right|^2.
\end{aligned}$$

As for the second term in (3.15), we have

$$\begin{aligned}&\left|\int_{\mathcal{R}} \zeta\left(k_{222} v^4+k_{22} 3 v^2 v_y\right)\left(\zeta v_y^{2\beta-1}\right)\right| \\ &\quad\leq C\|\xi v\|_{L^\infty}^4\left(\int_{\mathcal{R}}\left|\zeta v_y^\beta\right|^2\right)^{\frac{2\beta-1}{2\beta}}|\mathcal{R}|^{\frac{1}{2\beta}}+C\|\xi v\|_{L^\infty}^2 \int_{\mathcal{R}}\left|\zeta v_y^\beta\right|^2 \\ &\quad\leq C\|\xi v\|_{L^\infty}^4\left(\int_{\mathcal{R}}\left|\zeta v_y^\beta\right|^2+|\mathcal{R}|\right)+C\|\xi v\|_{L^\infty}^2 \int_{\mathcal{R}}\left|\zeta v_y^\beta\right|^2 .\end{aligned}$$

The remaining terms in (3.14) are handled similarly. Indeed, from (3.12), we see that the only differences in the remaining terms are that some powers of v are replaced by the same or smaller powers of u, y-derivatives by x-derivatives, and partial derivatives of k by others of the same or smaller order. This completes the proof of the lemma.

Theorem 3.6. *Suppose that w solves (3.9) so that with $u=w_x$ and $v=w_y$, the four functions u_x,u_y,v_x,v_y give a smooth solution of the system (3.11) in $\mathcal{R}$. Then for $q>2$, we have $u_x,u_y,v_x,v_y\in L^q$, i.e. $\nabla^2 w\in L^q$ with control provided R_1 is sufficiently small, depending on q.*

Proof. Using the one-dimensional Poincaré inequality, we have for $\beta>1$,

$$\begin{aligned}&\int_{\mathcal{R}}\left\{\left|\zeta u_x^\beta\right|^2+\left|\zeta u_y^\beta\right|^2+\left|\zeta v_x^\beta\right|^2+\left|\zeta v_y^\beta\right|^2\right\} \\ &\quad\leq C R_1^2 \int_{\mathcal{R}}\left\{\left|\partial_x \zeta u_x^\beta\right|^2+\left|\partial_x \zeta u_y^\beta\right|^2+\left|\partial_x \zeta v_x^\beta\right|^2+\left|\partial_x \zeta v_y^\beta\right|^2\right\} \\ &\quad\leq C R_1^2 \int_{\mathcal{R}}\left\{\left|\zeta \partial_x u_x^\beta\right|^2+\left|\zeta \partial_x u_y^\beta\right|^2+\left|\zeta \partial_x v_x^\beta\right|^2+\left|\zeta \partial_x v_y^\beta\right|^2\right\} \\ &\qquad+C A^2 R_1^2 \int_{\mathcal{R}}\left\{\left|\varrho_1 u_x^\beta\right|^2+\left|\varrho_1 u_y^\beta\right|^2+\left|\varrho_1 v_x^\beta\right|^2+\left|\varrho_1 v_y^\beta\right|^2\right\},\end{aligned}$$

since $|\partial_x\zeta|\leq A\varrho_1$. Now using the above lemma on the first term on the right side above, and then absorbing the term

$$C R_1^2 \mathcal{C}_1\left(\beta, \frac{1}{\beta-\frac{1}{2}}, B,\|\xi \nabla w\|_{\infty}\right) \int_{\mathcal{R}}\left\{\left|\zeta u_x^\beta\right|^2+\left|\zeta u_y^\beta\right|^2+\left|\zeta v_x^\beta\right|^2+\left|\zeta v_y^\beta\right|^2\right\}$$

into the left side for R_1 sufficiently small, we have

$$\begin{aligned}&\int_{\mathcal{R}}\left\{\left|\zeta u_x^\beta\right|^2+\left|\zeta u_y^\beta\right|^2+\left|\zeta v_x^\beta\right|^2+\left|\zeta v_y^\beta\right|^2\right\} \\ &\quad\leq C A^2\left(\frac{\beta}{2\beta-1}\right)^2 \int_{\mathcal{R}}\left\{\left|\varrho_1 u_x^\beta\right|^2+\left|\varrho_1 u_y^\beta\right|^2+\left|\varrho_1 v_x^\beta\right|^2+\left|\varrho_1 v_y^\beta\right|^2\right\} \\ &\qquad+C \mathcal{C}_3\left(\beta, \frac{1}{\beta-\frac{1}{2}},\|\xi \nabla w\|_{\infty}\right)+C \mathcal{C}_2\left(\beta, \frac{1}{\beta-\frac{1}{2}}, A, B,\|\xi \nabla w\|_{\infty}\right) \\ &\qquad\times \int_{\mathcal{R}} k\left\{\left|\varrho_2 u_x^\beta\right|^2+\left|\varrho_2 u_y^\beta\right|^2+\left|\varrho_2 v_x^\beta\right|^2+\left|\varrho_2 v_y^\beta\right|^2\right\}.\end{aligned}\tag{3.16}$$

Now the first term on the right is *under control* by our assumption that $\mathcal{L}$ is elliptic on the support of ϱ_1 (see Proposition 3.3 above). The second term is clearly *under* *control*. We will next show that the third term on the right hand side above is dominated by

$$\mathcal{C}\left(\beta,\frac{1}{\beta-\frac{1}{2}},A,B,\|\xi\nabla w\|_\infty\right)$$
$$\times\left\{1+\left\|\xi u_x^\beta\right\|_{L^p}^2+\left\|\xi u_y^\beta\right\|_{L^p}^2+\left\|\xi v_x^\beta\right\|_{L^p}^2+\left\|\xi v_y^\beta\right\|_{L^p}^2\right\}$$
$$\times\left\{1+\|\varkappa u_x\|_{L^2}^2+\|\varkappa u_y\|_{L^2}^2+\|\varkappa v_x\|_{L^2}^2+\|\varkappa v_y\|_{L^2}^2\right\},$$

for some $p<2$. Recall that we extend the usual convention regarding constants C to the functions $\mathcal{C}\left(\beta,\frac{1}{\beta-\frac{1}{2}},A,B,\|\xi\nabla w\|_\infty\right)$ - they may change from line to line, while remaining increasing in each variable separately. Indeed, with this done, we can then choose $\beta=\frac{2}{p}$ and conclude that $\nabla u,\nabla v\in L^q$ *with control* for $q=\frac{4}{p}>2$. In fact, we can continue to iterate this inequality as long as R_1^2 is sufficiently small. Thus we end up with $\nabla u,\nabla v\in L^q$ *with control* for q large provided R_1 is small enough.

The terms involving ϱ_2, namely

$$\int_{\mathcal{R}}k\left\{\left|\varrho_2u_x^\beta\right|^2+\left|\varrho_2u_y^\beta\right|^2+\left|\varrho_2v_x^\beta\right|^2+\left|\varrho_2v_y^\beta\right|^2\right\}, \tag{3.17}$$

are handled with Lemma 2.10 as follows. For $\beta>1$,

$$\begin{aligned}\int_{\mathcal{R}}k\left|\varrho_2v_y^\beta\right|^2dxdy&\leq C\beta\left|\int_{\mathcal{R}}\left(\xi I_1\varrho_2v_y^{\beta-1}\mathcal{L}v_y\right)\left(\xi I_1\varrho_2v_y^\beta\right)\right|\\&+C\frac{\beta}{\beta-1}\left|\int_{\mathcal{R}}\left(\sqrt{\varrho_2}v_y^{\frac{\beta}{2}-1}\mathcal{L}v_y\right)\left(\sqrt{\varrho_2}v_y^{\frac{\beta}{2}}\right)\right|^2\\&+C\left(p,\beta,A,k\right)^2\left(\int_{\mathcal{R}}\left|\xi v_y^\beta\right|^p\right)^{\frac{2}{p}}\\&=C\beta I+C\frac{\beta}{\beta-1}II+C\left(p,\beta,A,k\right)^2\left(\int_{\mathcal{R}}\left|\xi v_y^\beta\right|^p\right)^{\frac{2}{p}},\end{aligned} \tag{3.18}$$

where

$$C\left(p,\beta,A,k\right)=C_p\left(\sqrt{\beta\left(\beta-1\right)}+\frac{A^2\sqrt{\beta}}{\left(\beta-1\right)^{\frac{3}{2}}}+\|\varkappa k\|_{C^\nu}+A\left\|\varkappa\partial_yk\right\|_\infty+A^2\right),$$

and with similar estimates for u_x,u_y,v_x in place of v_y. We remind the reader of the Convention in the introduction: k means $k\left(x,w\left(x,y\right)\right)$ and ∂_yk means $k_2\left(x,w\left(x,y\right)\right)w_y$. We thus note that both $\|\varkappa k\|_{C^\nu}$ and $\|\varkappa k_y\|_\infty$ are *under control*. It follows that the last term on the right side of (3.18) has the desired form.

We first consider the simpler term II, and plugging in the nonlinear term for $\mathcal{L}v_y$, we have

$$\begin{aligned}\sqrt{II}&\leq\left|\int_{\mathcal{R}}\left(\sqrt{\varrho_2}v_y^{\frac{\beta}{2}-1}\left(\left(k_2\partial_y\right)^t3vv_y\right)\right)\left(\sqrt{\varrho_2}v_y^{\frac{\beta}{2}}\right)\right|\\&+\left|\int_{\mathcal{R}}\left(\sqrt{\varrho_2}v_y^{\frac{\beta}{2}-1}\left(k_{222}v^4+k_{22}3v^2v_y\right)\right)\left(\sqrt{\varrho_2}v_y^{\frac{\beta}{2}}\right)\right|\\&=III+IV.\end{aligned}$$

Now using limiting arguments to justify the needed formal manipulations (recall that $\beta > 1$ here), we have

$$\begin{aligned} III &= \left|\int_{\mathcal{R}} (3vv_y)\left(k_2\partial_y\varrho_2 v_y^{\beta-1}\right)\right| \leq C\left|\int_{\mathcal{R}} \left(3vv_y^{\frac{\beta}{2}}\right)\left(k_2\varrho_2\partial_y v_y^{\frac{\beta}{2}}\right)\right| + CA\left\|\xi v\right\|_{L^\infty}\int_{\mathcal{R}}\left|\xi v_y^\beta\right| \\ &\leq C\tilde{B}^2\int_{\mathcal{R}} k\left|\varrho_2\partial_y v_y^{\frac{\beta}{2}}\right|^2 + CA\left(\left\|\xi v\right\|_{L^\infty} + \left\|\xi v\right\|_{L^\infty}^2\right)\int_{\mathcal{R}}\left|\xi v_y^\beta\right| \\ &\leq \mathcal{C}\left(\beta, \frac{1}{\beta-\frac{1}{2}}, A, B, \left\|\varkappa\nabla w\right\|_\infty\right)\left(1 + \int_{\mathcal{R}}\left\{\left|\xi u_x^{\frac{\beta}{2}}\right|^2 + \left|\xi u_y^{\frac{\beta}{2}}\right|^2 + \left|\xi v_x^{\frac{\beta}{2}}\right|^2 + \left|\xi v_y^{\frac{\beta}{2}}\right|^2\right\}\right) \end{aligned}$$

by (3.13) with $\frac{\beta}{2}$ in place of β, and with ϱ_2 in place of ζ, upon combining all three integrals there under the common cutoff function ξ. This shows that term III is dominated by (**??**) with $p = 1$ upon using $\int_{\mathcal{R}} f \leq 1 + \left(\int_{\mathcal{R}} f\right)^2$, valid for any $f \geq 0$. The estimate for IV is

$$IV \leq C\left\|\xi v\right\|_{L^\infty}^4\left(\int_{\mathcal{R}}\left|\xi v_y^\beta\right| + |\mathcal{R}|\right) + C\left\|\xi v\right\|_{L^\infty}^2\int_{\mathcal{R}}\left|\xi v_y^\beta\right|.$$

Turning now to term I, and plugging in the nonlinear term for $\mathcal{L}v_y$, we have

$$\begin{aligned} I &\leq C\left|\int_{\mathcal{R}}\left\{\xi I_1\varrho_2 v_y^{\beta-1}\left((k_2\partial_y)^t\, 3vv_y\right)\right\}\left(\xi I_1\varrho_2 v_y^\beta\right)\right| \\ &+C\left|\int_{\mathcal{R}}\left\{\xi I_1\varrho_2 v_y^{\beta-1}\left(k_{222}v^4 + k_{22}3v^2v_y\right)\right\}\left(\xi I_1\varrho_2 v_y^\beta\right)\right| \\ &= V + VI. \end{aligned}$$

We can quickly dispense with term VI using that I_1 maps L^p to L^2 for $1 < p < 2$. We handle term V with the identity

$$v_y^{\beta-1}\partial_y k_2 vv_y = \frac{1}{\beta}\partial_y k_2 vv_y^\beta + \left(1 - \frac{1}{\beta}\right)v_y^\beta\partial_y k_2 v = \frac{1}{\beta}\partial_y k_2 vv_y^\beta + \left(1 - \frac{1}{\beta}\right)v_y^\beta\left(k_2v_y + k_{22}v^2\right)$$

to get

$$\begin{aligned} V &= C\left|\int_{\mathcal{R}}\left(\xi I_1\varrho_2 v_y^{\beta-1}\partial_y k_2 vv_y\right)\left(\xi I_1\varrho_2 v_y^\beta\right)\right| \\ &= C\left|\int_{\mathcal{R}}\left(\xi I_1\varrho_2\left[\frac{1}{\beta}\partial_y k_2 vv_y^\beta + \left(1-\frac{1}{\beta}\right)v_y^\beta\left(k_2v_y + k_{22}v^2\right)\right]\right)\left(\xi I_1\varrho_2 v_y^\beta\right)\right| \\ &\leq C\left|\int_{\mathcal{R}}\left(\xi I_1\varrho_2\partial_y k_2 vv_y^\beta\right)\left(\xi I_1\varrho_2 v_y^\beta\right)\right| + C\left|\int_{\mathcal{R}}\left(\xi I_1\varrho_2 v_y^\beta k_2 v_y\right)\left(\xi I_1\varrho_2 v_y^\beta\right)\right| \\ &+C\left|\int_{\mathcal{R}}\left(\xi I_1\varrho_2 v_y^\beta k_{22}v^2\right)\left(\xi I_1\varrho_2 v_y^\beta\right)\right| \\ &= VII + VIII + IX. \end{aligned}$$

For term VII, we commute ϱ_2 and ∂_y so that we can exploit the L^p boundedness of $I_1\partial_y$ as follows:

$$VII \leq C\left|\int_{\mathcal{R}}\left(\xi I_1\partial_y\varrho_2 k_2 vv_y^\beta\right)\left(\xi I_1\varrho_2 v_y^\beta\right)\right| + \left|\int_{\mathcal{R}}\left(\xi I_1\left(\partial_y\varrho_2\right)k_2 vv_y^\beta\right)\left(\xi I_1\varrho_2 v_y^\beta\right)\right|.$$

The first integral is

$$\begin{aligned}\left|\int_{\mathcal{R}}\left(I_{\frac{1}{2}}\xi^2 I_1\partial_y\varrho_2 k_2 v v_y^\beta\right)\left(I_{\frac{1}{2}}\varrho_2 v_y^\beta\right)\right| &\leq C\int_{\mathcal{R}}\left|I_{\frac{1}{2}}\xi^2 I_1\partial_y\varrho_2 k_2 v v_y^\beta\right|^2 + C\int_{\mathcal{R}}\left|I_{\frac{1}{2}}\varrho_2 v_y^\beta\right|^2\\ &\leq C\left(\int_{\mathcal{R}}\left|\xi^2 I_1\partial_y\varrho_2 k_2 v v_y^\beta\right|^p\right)^{\frac{2}{p}} + C\left(\int_{\mathcal{R}}\left|\varrho_2 v_y^\beta\right|^p\right)^{\frac{2}{p}}\\ &\leq C\left(1+\|\xi v\|_{L^\infty}^2\right)\left(\int_{\mathcal{R}}\left|\xi v_y^\beta\right|^p\right)^{\frac{2}{p}},\end{aligned}$$

where $\frac{1}{2}=\frac{1}{p}-\frac{1}{4}$, and the second is dominated by

$$C\int_{\mathcal{R}}\left|\xi I_1\left(\partial_y\varrho_2\right)k_2 v v_y^\beta\right|^2 + C\int_{\mathcal{R}}\left|\xi I_1\varrho_2 v_y^\beta\right|^2 \leq C\left(1+A^2\|\xi v\|_{L^\infty}^2\right)\left(\int_{\mathcal{R}}\left|\xi v_y^\beta\right|^p\right)^{\frac{2}{p}},$$

for any $1<p<2$. Term IX is dominated by

$$C\int_{\mathcal{R}}\left|\xi I_1\varrho_2 v_y^\beta k_{22}v^2\right|^2 + C\int_{\mathcal{R}}\left|\xi I_1\varrho_2 v_y^\beta\right|^2 \leq C\left(1+\|\xi v\|_{L^\infty}^4\right)\left(\int_{\mathcal{R}}\left|\xi v_y^\beta\right|^p\right)^{\frac{2}{p}},$$

for any $1<p<2$ also.

In term $VIII$, the most problematic, we have an additional power of v_y to deal with. We write using $\varrho_2=\varrho_2\xi$,

$$\begin{aligned}VIII &= C\left|\int_{\mathcal{R}}\left(\varrho_2 k_2 v_y^{\beta+1}\right)\left(I_1\xi^2 I_1\varrho_2 v_y^\beta\right)\right|\\ &\leq C\left\|\xi I_1\xi^2 I_1\varrho_2 v_y^\beta\right\|_{L^\infty}\int_{\mathcal{R}}\left|v_y k_2\varrho_2 v_y^\beta\right|\\ &\leq C\left(\int_{\mathcal{R}}\left|\xi v_y^\beta\right|^p\right)^{\frac{1}{p}}\left\{\frac{C}{\varepsilon}\int_{\mathcal{R}}\left|\xi v_y\right|^2+\tilde{B}^2\varepsilon\int_{\mathcal{R}}k\left|\varrho_2 v_y^\beta\right|^2\right\},\end{aligned}$$

for any $1<p<2$ (since $I_2: L^p_{compact}\to L^\infty_{loc}$ for such p) and $\varepsilon>0$. We can choose $\varepsilon>0$ sufficiently small, in fact $\varepsilon\approx\tilde{B}^{-2}\left(\int_{\mathcal{R}}\left|\xi v_y^\beta\right|^p\right)^{-\frac{1}{p}}$, so that the term $\left(\int_{\mathcal{R}}\left|\xi v_y^\beta\right|^p\right)^{\frac{1}{p}}\varepsilon\tilde{B}^2\int_{\mathcal{R}}k\left|\varrho_2 v_y^\beta\right|^2$ can be absorbed into the left side of (3.18). Then term $VIII$ is dominated by

$$\left(\int_{\mathcal{R}}\left|\xi v_y^\beta\right|^p\right)^{\frac{1}{p}}\frac{C}{\varepsilon}\int_{\mathcal{R}}\left|\xi v_y\right|^2\approx C\tilde{B}^2\left(\int_{\mathcal{R}}\left|\xi v_y^\beta\right|^p\right)^{\frac{2}{p}}\int_{\mathcal{R}}\left|\xi v_y\right|^2,$$

as required. The remaining terms in (3.17) are handled similarly and this completes the proof of the theorem.

3.3. **The iteration.** We can now obtain our *a priori* inequality (3.1) for the quasilinear equation (1.1) We briefly restate Theorem 1.1 as follows.

Theorem 3.7. *Suppose that k is smooth and positive for $x\neq 0$. Then with $\mathcal{C}_\alpha$ as in Theorem 1.1, we have*

$$\left\|\zeta D^\alpha w\right\|_\infty\leq\mathcal{C}_\alpha\left(\left\|\varkappa\nabla w\right\|_\infty, L\right),\quad|\alpha|\geq 0.$$

for all smooth solutions w of (1.1) in Ω' such that $(x,w(x,y))\in L$ for all (x,y) in the support of $\varkappa$.

Proof. Recall that in the subsection on "Reverse Sobolev Embedding", we used the fact that $\nabla w \in L^{\infty}$ *with control* together with the fact that $\nabla w = (w_x, w_y)$ satisfies the system (3.4) to conclude that $\nabla^2 w \in L^2$ *with control*. We now wish to deduce that $\nabla^3 w \in L^2$ *with control* from a somewhat weaker integrability assumption on $\nabla^2 w$, plus the fact that $\nabla^2 w$ satisfies an appropriate system. We continue to use the phrase "*under control*" to mean bounded by an increasing function $\mathcal{C}_\alpha\left(\|\varkappa \nabla w\|_\infty, L\right)$ of $\|\varkappa \nabla w\|_\infty \in [0,\infty)$ and $L \in \mathcal{P}_c(\Omega)$, etc. At this point, we already know that if w is a smooth solution of (1.1) in $\mathcal{R}$, then

$$\begin{aligned} \nabla^2 w &\in L^q \text{ with control}, \quad \text{for } q \text{ large depending on } R_1, \\ \nabla w &\in L^\infty \text{ with control}, \end{aligned}$$

and that $\nabla^2 w$ satisfies (3.12), a system of the form

$$\mathcal{L}\left(\nabla^2 w\right) = P(\nabla w)\left(\nabla^2 w\right) + Q(\nabla w) + T^t\left[R(\nabla w)\left(\nabla^2 w\right) + S(\nabla w)\right], \tag{3.19}$$

where $P(\nabla w)$, $Q(\nabla w)$, $R(\nabla w)$ and $S(\nabla w)$ (at this early stage in the iteration, the polynomial S vanishes) are polynomials in the components of the bounded vector field ∇w with partial derivatives of k as coefficients. Also, the expression $P(\nabla w)\left(\nabla^2 w\right)$ means sums of such polynomials times some second order derivatives of w. Finally, T is a subunit vector field of the form $k_i \partial_y$. We can now apply the methods of the subsection "Reverse Sobolev embedding", since the components of $\nabla^2 w$ appear only to the first power multiplied by components of ∇w, which are bounded. To estimate the L^2_{loc} norm of $\nabla^3 w$ by the technique of the proof of Theorem 3.4, we need to estimate $\int_{\mathcal{R}} \left|\partial_x\left(\zeta \partial \eta \nabla^2 w\right)\right|^2$. To do this , we will use the analogue of Lemma 3.2 to estimate

$$\int_{\mathcal{R}} \left(\left|\partial_x\left(\zeta \partial \eta \nabla^2 w\right)\right|^2 + k\left|\partial_y\left(\zeta \partial \eta \nabla^2 w\right)\right|^2\right) \tag{3.20}$$

for $\partial = \partial_x$ or $\partial = \partial_y$ by applying Corollary 2.7. The main terms to be estimated are of the form

$$\int_{\mathcal{R}} \left(\zeta \partial \eta \mathcal{L} \nabla^2 w\right)\left(\zeta \partial \eta \nabla^2 w\right). \tag{3.21}$$

Replacing $\mathcal{L}\nabla^2 w$ by one of the terms in (3.19), say $T^t\left[R(\nabla w)\left(\nabla^2 w\right)\right]$, we can decompose the resulting expression into three pieces $I + II + III$ as in the proof of Lemma 3.2. Term I has the form

$$\int_{\mathcal{R}} \left(\zeta \partial \eta R(\nabla w)\left(\nabla^2 w\right)\right)\left(k_j \partial_y \zeta \partial \eta \nabla^2 w\right), \tag{3.22}$$

which is dominated by

$$\frac{C}{\alpha}\int_{\mathcal{R}} \left|\zeta \partial \eta R(\nabla w)\left(\nabla^2 w\right)\right|^2 + C\alpha \tilde{B}^2 \int_{\mathcal{R}} k\left|\partial_y \zeta \partial \eta \nabla^2 w\right|^2.$$

Now the second term here can be absorbed, while in the first term we use

$$\partial R(\nabla w)\,\eta\left(\nabla^2 w\right) = R(\nabla w)\,\partial \eta\left(\nabla^2 w\right) + \eta \nabla^2 w\left(\partial R(\nabla w)\right).$$

Now $R(\nabla w)$ is bounded, and so we can use the one-dimensional Poincaré inequality to get

$$\int_{\mathcal{R}} \left|R(\nabla w)\,\partial \eta\left(\nabla^2 w\right)\right|^2 \le C R_1^2 \left\|\xi R(\nabla w)\right\|_{L^\infty}^2 \int_{\mathcal{R}} \left|\partial_x \zeta \partial \eta\left(\nabla^2 w\right)\right|^2,$$

which can be absorbed for R_1 small enough. Finally, $\partial R(\nabla w)$ consists of bounded terms times components of $\nabla^2 w$, plus bounded terms, and we simply use that $\nabla^2 w \in L^4$ *with control*. Terms II and III are also handled just as in Lemma 3.2.

The result of all this is that $\nabla^2 w \in H^1_{loc}$ *with control*, or $\nabla^3 w \in L^2_{loc}$ *with control*. Moreover, $\nabla^3 w$ solves a system of equations obtained by differentiating (3.12), and thus has the form

$$
\begin{aligned}
(3.23) \qquad -\mathcal{L}\left(\nabla^3 w\right) &= (k_i \partial_y)^t \left\{\left(\nabla^2 w\right)^2 + (\nabla w)\left(\nabla^3 w\right)\right\} + \mathcal{K}_4 (\nabla w)^5 \\
&\quad + \mathcal{K}_3 (\nabla w)^3 \left(\nabla^2 w\right) + \mathcal{K}_2 (\nabla w)\left(\nabla^2 w\right)^2 \\
&\quad + \mathcal{K}_2 (\nabla w)^2 \left(\nabla^3 w\right),
\end{aligned}
$$

where $\mathcal{K}_j$ denotes a derivative of k of order j, and $\mathcal{K}_j (\nabla w)^m$ represents a sum of products of such derivatives times m^{th} order products of first order derivatives of w. For example, $v_{yy} = w_{yyy}$ satisfies

$$
-\partial_y \mathcal{L} v_y = \partial_y \left[(k_2 \partial_y)^t \, 3 v v_y + \left\{k_{222} v^4 + k_{22} 3 v^2 v_y\right\}\right],
$$

or

$$
\begin{aligned}
-\mathcal{L} v_{yy} &= (k_2 \partial_y)^t \left(3 v_y^2 + 4 v v_{yy}\right) + k_{222} 3 v^3 v_y + k_{22}\left(6 v v_y^2 + 3 v^2 v_{yy}\right) \\
&\quad + \left\{k_{2222} v^5 + k_{222} 4 v^3 v_y + k_{222} 3 v^3 v_y + k_{22} 3\left(2 v v_y^2 + v^2 v_{yy}\right)\right\}.
\end{aligned}
$$

Here $\mathcal{K}_4 = k_{2222}$, etc.

Note that this system has the form

$$
\mathcal{L}\left(\nabla^3 w\right) = P(\nabla w)\left(\nabla^3 w\right) + Q\left(\nabla w, \nabla^2 w\right) + T^t\left[R(\nabla w)\left(\nabla^3 w\right) + S\left(\nabla w, \nabla^2 w\right)\right],
$$

where P, Q, R and S are polynomials with partial derivatives of k as coefficients. Altogether, we have

$$
\begin{aligned}
&\nabla w, \nabla^2 w, \nabla^3 w \in L^2 \textit{ with control}, \\
&\mathcal{L}\left(\nabla^3 w\right) = P(\nabla w)\left(\nabla^3 w\right) + Q\left(\nabla w, \nabla^2 w\right) + T^t\left[R(\nabla w)\left(\nabla^3 w\right) + S\left(\nabla w, \nabla^2 w\right)\right].
\end{aligned}
$$

Note that the Sobolev embedding theorem shows that we actually have $\nabla^2 w \in L^q$, for *all* $q < \infty$ (prior to this we only had $\nabla^2 w \in L^q$ for q depending on R_1) and $\nabla w \in L^\infty$ *with control* (the latter assertion is of course redundant at this point). We can now apply the methods of the previous subsection "An L^p improvement", since the unknowns $\nabla^3 w$ appear only to the first power and times bounded terms consisting of polynomials in ∇w, so that we can use $\nabla^3 w \in L^2$ *with control*. Terms of $\nabla^2 w$ can appear to higher powers (actually, at most squared, which means we need only $q = 4$), but they can be handled since $\nabla^2 w \in L^q$ *with control* for $q < \infty$. The result here is that $\nabla^3 w \in L^q$ *with control* for q large depending on R_1, and so also $\nabla^2 w \in L^\infty$ *with control*, by the Sobolev embedding theorem. We can now apply the methods of the subsection "Reverse Sobolev embedding" as we did just above, and the result is that $\nabla^4 w \in L^2$ *with control*. Finally, computing $\mathcal{L}\left(\nabla^4 w\right)$, we obtain

$$
\begin{aligned}
\nabla w, \nabla^2 w, \nabla^3 w, \nabla^4 w &\in L^2 \textit{ with control}, \\
\mathcal{L}\left(\nabla^4 w\right) &= P(\nabla w)\left(\nabla^4 w\right) + Q\left(\nabla w, \nabla^2 w\right)\left(\nabla^3 w\right) + Q_0\left(\nabla w, \nabla^2 w\right) \\
&\quad + T^t\left[R(\nabla w)\left(\nabla^4 w\right) + S\left(\nabla w, \nabla^2 w\right)\left(\nabla^3 w\right) + S_0\left(\nabla w, \nabla^2 w\right)\right],
\end{aligned}
$$

where again by the Sobolev embedding theorem, $\nabla^3 w \in L^q$, $q < \infty$ and $\nabla^2 w \in L^\infty$ *with control*. Note that this time, components of both $\nabla^4 w$ and $\nabla^3 w$ appear only to the first power, multiplied by polynomials in the components of $\nabla^2 w$ and ∇w, which are bounded. This is the sense in which the equations for higher order derivatives become progressively less nonlinear.

We can now iterate this process to obtain

$$\begin{aligned}
(3.24) \qquad \nabla^j w &\in L^2 \textit{ with control}, \quad 1 \le j \le \ell + 1. \\
\mathcal{L}\left(\nabla^{\ell+1} w\right) &= P(\nabla w)\left(\nabla^{\ell+1} w\right) + Q\left(\nabla w, \nabla^2 w\right)\left(\nabla^\ell w\right) \\
&\quad + Q_0\left(\nabla w, ..., \nabla^{\ell-1} w\right) + T^t\left[R(\nabla w)\left(\nabla^{\ell+1} w\right)\right] \\
&\quad + T^t\left[S\left(\nabla w, \nabla^2 w\right)\left(\nabla^\ell w\right) + S_0\left(\nabla w, ..., \nabla^{\ell-1} w\right)\right].
\end{aligned}$$

for all ℓ by induction on ℓ, where P, Q, Q_0, R, S, S_0 are polynomials with partial derivatives of k as coefficients, and as before, the Sobolev embedding theorem shows that the first line in 3.24 can be improved to

$$\begin{aligned}
\nabla^j w &\in L^\infty \textit{ with control}, \quad 1 \le j \le \ell - 1, \\
\nabla^\ell w &\in L^q \textit{ with control}, \quad \textit{for } q < \infty, \\
\nabla^{\ell+1} w &\in L^2 \textit{ with control}.
\end{aligned}$$

We emphasize that $\nabla^{\ell+1} w$ and $\nabla^\ell w$ appear linearly in (3.24) with coefficients involving derivatives of order at most two of w, and that $\nabla^{\ell-1} w$ and earlier derivatives are bounded. For example, although we will not need the following information on the form of $\mathcal{L}\left(\partial_y^\ell v\right)$, it turns out that $\mathcal{L}\left(\partial_y^\ell v\right)$ is a sum of terms of the type that arise from $\mathcal{P} = \sum_{j=0}^{\ell}\left(\partial_2^{\ell+2-j} k\right)\left(\partial_y^j v^{\ell+3-j}\right)$ upon expanding $\partial_y^j v^{\ell+3-j}$. More specifically, we mean that the relation between derivatives of v and derivatives of k in the expansion of $\mathcal{L}\left(\partial_y^\ell v\right)$ is the same as in the expansion of $\mathcal{P}$. As a consequence, $\partial_y^i v$ appears linearly in $\mathcal{L}\left(\partial_y^\ell v\right)$ and $\mathcal{P}$ for $i > \frac{\ell}{2}$.

Returning to the induction, if (3.24) holds for a given ℓ, then as above, the previous subsection "An L^p improvement" shows that $\nabla^{\ell+1} w \in L^q$ *with control*, for q large, and so by the Sobolev embedding theorem that $\nabla^\ell w \in L^\infty$. The subsection "Reverse Sobolev embedding" then shows that $\nabla^{\ell+2} w \in L^2$. It is in these iterations that we require R_1 to be successively smaller as the constants involving earlier derivatives become progressively larger. Differentiating the equation for $\mathcal{L}\left(\nabla^{\ell+1} w\right)$ yields the same form for $\mathcal{L}\left(\nabla^{\ell+2} w\right)$. This establishes (3.24) for $\ell+1$ and completes the proof of the *a priori* estimates (3.1).

We remark that for $\ell \ge 4$, the technique of the section "Reverse Sobolev embedding" only requires (3.24) in order to conclude $\nabla^{\ell+2} w \in L^2$ *with control*, rather than having to first use the Moser iteration to obtain $\nabla^{\ell+1} w \in L^q$, for q large. As a result, we can inductively prove (3.24) for $\ell \ge 5$ (assuming it holds for $\ell = 4$) without resorting to the Moser iteration techniques of the section "An L^p improvement". To illustrate, we estimate the analogues of (3.20) and(3.21) with $\nabla^2 w$ replaced by $\nabla^{\ell+1} w$:

$$(3.25) \qquad \int_{\mathcal{R}}\left(\left|\partial_x\left(\zeta \partial \eta \nabla^{\ell+1} w\right)\right|^2 + k\left|\partial_y\left(\zeta \partial \eta \nabla^{\ell+1} w\right)\right|^2\right)$$

and

$$\int_{\mathcal{R}} \left(\zeta \partial \eta \mathcal{L} \nabla^{\ell+1} w\right) \left(\zeta \partial \eta \nabla^{\ell+1} w\right). \tag{3.26}$$

After plugging into (3.26) part of the formula for $\mathcal{L}\left(\nabla^{\ell+1} w\right)$ in (3.24), namely

$$T^t \left[R(\nabla w)\left(\nabla^{\ell+1} w\right) + S\left(\nabla w, \nabla^2 w\right)\left(\nabla^{\ell} w\right)\right],$$

and then moving T^t to the other side of the integral, we obtain the following analogue of term I in (3.22):

$$\int_{\mathcal{R}} \left(\zeta \partial \eta R(\nabla w)\left(\nabla^{\ell+1} w\right)\right)\left(k_j \partial_y \zeta \partial \eta \nabla^{\ell+1} w\right) \\ + \int_{\mathcal{R}} \left(\zeta \partial \eta S\left(\nabla w, \nabla^2 w\right)\left(\nabla^{\ell} w\right)\right)\left(k_j \partial_y \zeta \partial \eta \nabla^{\ell+1} w\right).$$

The more problematic term is the second one which can be dominated by

$$\frac{C}{\alpha} \int_{\mathcal{R}} \left|\zeta \partial \eta S\left(\nabla w, \nabla^2 w\right)\left(\nabla^{\ell} w\right)\right|^2 + C \alpha \tilde{B}^2 \int_{\mathcal{R}} k \left|\partial_y \zeta \partial \eta \nabla^{\ell+1} w\right|^2.$$

The second term here can be absorbed into (3.25), while for the first we use

$$\partial S\left(\nabla w, \nabla^2 w\right) \eta \left(\nabla^{\ell} w\right) = S\left(\nabla w, \nabla^2 w\right) \partial \eta \left(\nabla^{\ell} w\right) + \eta \left(\nabla^{\ell} w\right) \partial S\left(\nabla w, \nabla^2 w\right).$$

Since $S\left(\nabla w, \nabla^2 w\right)$ is bounded for $\ell \geq 3$, we can use that $\int_{\mathcal{R}} \left|\zeta \partial \eta \left(\nabla^{\ell} w\right)\right|^2$ is *under control* by induction to handle the L^2 norm of the first term here. As for the second, $\partial S\left(\nabla w, \nabla^2 w\right)$ includes components of $\nabla^3 w$, which will be bounded provided $\ell \geq 4$. The remaining terms are also handled by such techniques.

4. Proofs of the main theorems

In this final section, we apply Theorem 3.7, our *a priori* estimates in terms of the gradient, to obtain the remaining theorems mentioned at the beginning of the paper.

4.1. Close to one variable curvature.

We begin by proving Theorem **??**. Recall that the desired conclusion is

$$\left\|\zeta D^{\alpha} w\right\|_{\infty} \leq \mathcal{C}_{\alpha}(L), \tag{4.1}$$

where w is a solution of the quasilinear equation (3.2) satisfying the condition (3.3),

$$(x, w(x,y)) \in L \textit{ for all } (x,y) \in \textit{support}(\varkappa).$$

We will say that an expression involving derivatives of w is *under special control* if it is dominated by the right side of (4.1). Note that this is a stronger condition than requiring that w is *under control*. Since $|u| \leq C\sqrt{v}$ by our assumption in (1.12), it is enough by the previous theorem, Theorem 1.1, to prove that $\|\zeta v\|_\infty \leq \mathcal{C}_\alpha(L)$. This will be accomplished by using Plancherel's theorem in the following way:

$$\begin{aligned}
\|\zeta v\|_{L^\infty} &\leq \left\|\widehat{\zeta v}\right\|_{L^1} \leq \left\{\int\int \frac{d\sigma d\tau}{1+|\sigma|^2+|\tau|^2+|\sigma|^2|\tau|^2}\right\}^{\frac{1}{2}} \times \\
&\quad \left\{\int\int \left(1+|\sigma|^2+|\tau|^2+|\sigma|^2|\tau|^2\right)\left|\widehat{\zeta v}(\sigma,\tau)\right|^2 d\sigma d\tau\right\}^{\frac{1}{2}} \\
&\leq C\left\{\int\int\left(|\zeta v|^2+|(\zeta v)_x|^2+\left|(\zeta v)_y\right|^2+\left|(\zeta v)_{xy}\right|^2\right)dxdy\right\}^{\frac{1}{2}}.
\end{aligned}$$

This calculation reduces matters to showing that

$$\|\zeta v\|_{L^2}, \|\nabla(\zeta v)\|_{L^2}, \|\partial_x\partial_y(\zeta v)\|_{L^2}, \tag{4.2}$$

are all *under special control*. This in turn will be accomplished by establishing in succession that the following k-gradient integrals are *under special control*. Here, and often in subsequent inequalities, the cutoff functions may change from instance to instance:

$$\begin{aligned}
&\int_{\mathcal{R}}\left(|\zeta u|^2+k|\zeta v|^2\right) \quad \textit{is under special control,} \\
&\int_{\mathcal{R}}\left(|\zeta\partial_x v|^2+k|\zeta\partial_y v|^2\right) \quad \textit{is under special control,} \\
&\int_{\mathcal{R}}\left(|\zeta\partial_x\eta v_y|^2+k|\zeta\partial_y\eta v_y|^2\right) \quad \textit{is under special control.}
\end{aligned} \tag{4.3}$$

Indeed, Poincaré's inequality in one variable shows that the first term $\|\zeta v\|_{L^2}$ in (4.2) is controlled by $\|\zeta\partial_x v\|_{L^2}+\|\zeta_x v\|_{L^2}$. Now the term $\|\zeta\partial_x v\|_{L^2}$ is included in the second line of (4.3) while the other term $\|\zeta_x v\|_{L^2}$ is controlled using (1.12) and the first line of (4.3) since $k\geq c>0$ on the support of ζ_x.

The second term in (4.2) can be controlled, allowing for a change in cutoff functions as announced above and taking into account terms already estimated, by $\left\|\zeta_y v\right\|_{L^2}+\|\eta v_y\|_{L^2}$. The first of these is controlled by $\left\|\zeta_{xy}v\right\|_{L^2}+\left\|\zeta_y v_x\right\|_{L^2}$ by Poincaré's inequality, and both of these are controlled as above. Poincaré's inequality and earlier estimates again show that the second term, $\|\eta v_y\|_{L^2}$, is controlled by $\|\zeta\partial_x\eta v_y\|_{L^2}+\|\eta_x v_y\|_{L^2}$. The term $\|\zeta\partial_x\eta v_y\|_{L^2}^2$ is included in the third line of (4.3), while the term $\|\eta_x v_y\|_{L^2}$ is controlled by the second line of (4.3) since $k\geq c>0$ on the support of η_x.

The third term in (4.2) is controlled by $\|\zeta\partial_x\partial_y v\|_{L^2}+\|[\partial_x\partial_y,\zeta]v\|_{L^2}$. Now assuming here, as we may, that $\eta=1$ on the support of ζ, the first of these terms squared is included in the third line of (4.3). The second is controlled in terms of $\|\nabla(\zeta v)\|_{L^2}$ (for a cutoff function ζ with an enlarged support), which is the second term in (4.2) and has already been controlled in the previous paragraph.

We have from (2.5) with $k=k(x,w(x,y))$ and $\mathcal{L}=\partial_x^2+\partial_y k(x,w(x,y))\partial_y$ that

$$\int_{\mathcal{R}}\left(|\zeta\partial_x w|^2+\left|\zeta\sqrt{k}\partial_y w\right|^2\right)\leq -2\int_{\mathcal{R}}(\zeta\mathcal{L}w)(\zeta w)+4A^2\int_{\mathcal{R}}|\varrho_1 w|^2+4A^2\int_{\mathcal{R}}k|\varrho_2 w|^2.$$

Since $\mathcal{L}w=0$, we have $\|\zeta u\|_{L^2}+\left\|\zeta\sqrt{k}v\right\|_{L^2}\le C\,\|\xi w\|_\infty$, which by (3.3) proves the first assertion in (4.3). Now replacing w by v we obtain

(4.4)
$$\int_{\mathcal{R}}\left(|\zeta\partial_x v|^2+k\,|\zeta\partial_y v|^2\right)\le -2\int_{\mathcal{R}}(\zeta\mathcal{L}v)(\zeta v)+4A^2\int_{\mathcal{R}}|\varrho_1 v|^2+4A^2\int_{\mathcal{R}}k\,|\varrho_2 v|^2 .$$

Now $\int_{\mathcal{R}}|\varrho_1 v|^2\le C\int_{\mathcal{R}}k\,|\varrho_1 v|^2$ (since $k\ge c$ on the support of ϱ_1) and $\int_{\mathcal{R}}k\,|\varrho_j v|^2\le\left\|\xi\sqrt{k}v\right\|_{L^2}\le C\,\|\varkappa w\|_\infty$ by the previous inequality. Next, by (3.4),

(4.5)
$$\left|\int_{\mathcal{R}}(\zeta\mathcal{L}v)(\zeta v)\right|=\left|\int_{\mathcal{R}}\left(k_2v^2\right)\partial_y\left(\zeta^2 v\right)\right|\le\left|\int_{\mathcal{R}}\left(k_2v^2\right)\left(\zeta^2 v_y\right)\right|+\left|\int_{\mathcal{R}}\left(k_2v^2\right)\left(2\zeta\zeta_y v\right)\right|.$$

Now by our hypothesis (1.11), the second term is dominated by

$$C\int_{\mathcal{R}}\zeta\left|\zeta_y\right|k^{\frac{3}{2}}\,|v|^3\le C\int_{\mathcal{R}}\zeta\left|\zeta_y\right|k^{\frac{1}{2}}\,|v|^2$$

since $|kv|\le C$ by (1.12). Continuing, we bound the above by

$$\begin{aligned}C\int_{\mathcal{R}}\left|\zeta_y\right|^2k\,|v|^2+C\int_{\mathcal{R}}|\zeta v|^2&\le CA^2\int_{\mathcal{R}}k\,|\varrho_2 v|^2+CR_1^2\int_{\mathcal{R}}|\partial_x\zeta v|^2\\&\le CA^2\int_{\mathcal{R}}k\,|\varrho_2 v|^2+CR_1^2\int_{\mathcal{R}}|\zeta\partial_x v|^2+CR_1^2\int_{\mathcal{R}}|\zeta_x v|^2 .\end{aligned}$$

The first of these terms is dominated by $CA^2\,\|\varkappa w\|_\infty^2$ by the first inequality in (4.3). The second term on the right can be absorbed into the left side of (4.4) for R_1 sufficiently small, and the third is controlled since ζ_x is supported where $k\ge c>0$. Indeed, we then have $\int_{\mathcal{R}}|\zeta_x v|^2\le\frac{A^2}{c}\int_{\mathcal{R}}k\,|\xi v|^2$, which is *under special control* by the first line in (4.3).

The first term in (4.5) satisfies

$$\begin{aligned}\left|\int_{\mathcal{R}}\left(k_2v^2\right)\left(\zeta^2v_y\right)\right|&\le C\left|\int_{\mathcal{R}}\left(k^{\frac{1}{2}}\zeta v_y\right)\left(\zeta kv^2\right)\right|\le C\varepsilon\int_{\mathcal{R}}k\,|\zeta v_y|^2+\frac{C}{\varepsilon}\int_{\mathcal{R}}\left(\zeta^2k^2v^4\right)\\&\le C\varepsilon\int_{\mathcal{R}}k\,|\zeta v_y|^2+\frac{C}{\varepsilon}\int_{\mathcal{R}}(\zeta v)^2 ,\end{aligned}$$

by (1.12), where the term $C\varepsilon\int_{\mathcal{R}}\zeta k\,|v_y|^2$ can be absorbed on the left side of (4.4), and the remaining term is bounded by

$$\frac{C}{\varepsilon}R_1^2\int_{\mathcal{R}}|\partial_x\zeta v|^2\le\frac{C}{\varepsilon}R_1^2\int_{\mathcal{R}}|\zeta\partial_x v|^2+\frac{C}{\varepsilon}R_1^2\int_{\mathcal{R}}|\zeta_x v|^2 .$$

This can be handled as above, absorbing the first term on the right for R_1 sufficiently small, and using $k\ge c>0$ on the second term. This proves the second line in (4.3), and hence also $\|\zeta v\|_{L^2}\le C\,\|\varkappa w\|_\infty$ by the one-dimensional Poincaré inequality.

In preparation for proving the third line in (4.3), we will now use the Moser iteration technique to boost the integrability of v to $\|\zeta v\|_{L^6}\le C\,\|\varkappa w\|_\infty$. The inequality in Lemma 2.8 yields

$$\begin{aligned}\int_{\mathcal{R}}\left(\left|\zeta\partial_x v^\beta\right|^2+k\left|\zeta\partial_y v^\beta\right|^2\right)&\le\frac{2\beta^2}{2\beta-1}\left|\int_{\mathcal{R}}(\zeta\mathcal{L}v)\left(\zeta v^{2\beta-1}\right)\right|\\&+\left(\frac{2\beta}{2\beta-1}\right)^2A^2\int_{\mathcal{R}}\left|\varrho_1v^\beta\right|^2+\left(\frac{2\beta}{2\beta-1}\right)^2A^2\int_{\mathcal{R}}k\left|\varrho_2v^\beta\right|^2 ,\end{aligned}$$

for $\beta > \frac{1}{2}$. Now $k \geq c > 0$ on the support of ϱ_1, and so $\int_{\mathcal{R}} \left|\varrho_1 v^\beta\right|^2 \leq C \int_{\mathcal{R}} k \left|\varrho_1 v^\beta\right|^2$. As a result of this together with (1.12), the second and third terms on the right above are dominated by

$$C\left(\frac{\beta}{2\beta-1}\right)^2 A^2 \int_{\mathcal{R}} \left|\xi v^{\beta-\frac{1}{2}}\right|^2. \tag{4.6}$$

Remark 4.1. *Note that the inequality $kv \leq C$ from (1.12) has permitted us to avoid using the difficult Lemma 2.10 to handle the term $\int_{\mathcal{R}} k\left|\varrho_2 v^\beta\right|^2$.*

We will now show that the first term is bounded by a similar integral. We have

$$\begin{aligned} \left|\int_{\mathcal{R}} (\zeta\mathcal{L}v)\left(\zeta v^{2\beta-1}\right)\right| &= \left|\int_{\mathcal{R}} \left(\zeta\partial_y k_2 v^2\right)\left(\zeta v^{2\beta-1}\right)\right| \\ &\leq \left|\int_{\mathcal{R}} \left(\zeta k_2 v^2\right)\left(\zeta\partial_y v^{2\beta-1}\right)\right| + 2\left|\int_{\mathcal{R}} \left(\zeta_y k_2 v^2\right)\left(\zeta v^{2\beta-1}\right)\right|. \end{aligned} \tag{4.7}$$

Now the first integral here satisfies

$$\begin{aligned} &\left|\int_{\mathcal{R}} \left(\zeta k_2 v^2\right)\left(\zeta\partial_y v^{2\beta-1}\right)\right| \\ &= \frac{2\beta-1}{\beta}\left|\int_{\mathcal{R}} \left(\zeta k_2 v^{\beta+1}\right)\left(\zeta\partial_y v^\beta\right)\right| \leq C\frac{2\beta-1}{\beta}\int_{\mathcal{R}} \left|\zeta k v^{\beta+1}\right|\left|\sqrt{k}\zeta\partial_y v^\beta\right| \\ &\leq C\frac{2\beta-1}{\beta}\int_{\mathcal{R}} \left|\zeta v^\beta\right|\left|\sqrt{k}\zeta\partial_y v^\beta\right| \leq \frac{C}{\alpha}\int_{\mathcal{R}} \left|\zeta v^\beta\right|^2 + C\alpha\int_{\mathcal{R}} k\left|\zeta\partial_y v^\beta\right|^2, \end{aligned}$$

since $\frac{2\beta-1}{\beta} \leq 2$. The second term here can be absorbed for α chosen small enough, while by Poincaré's inequality in one variable, the first is dominated by

$$\frac{C}{\alpha}R_1^2\int_{\mathcal{R}} \left|\partial_x\zeta v^\beta\right|^2 \leq \frac{C}{\alpha}R_1^2\int_{\mathcal{R}} \left|\zeta\partial_x v^\beta\right|^2 + \frac{C}{\alpha}R_1^2\int_{\mathcal{R}} \left|\zeta_x v^\beta\right|^2.$$

The first integral on the right here can now be absorbed for R_1 small enough, and the second is at most $\frac{C}{\alpha}A^2R_1^2\int_{\mathcal{R}}\left|\xi v^{\beta-\frac{1}{2}}\right|^2$ since $v \leq \frac{C}{k} \leq \frac{C}{c}$ on the support of ζ_x. The second integral on the right side of (4.7) is at most

$$C\int_{\mathcal{R}} k^{\frac{3}{2}}\left|\zeta_y\zeta v^{2\beta+1}\right| \leq C\int_{\mathcal{R}} \left|\zeta_y\zeta v^{2\beta-\frac{1}{2}}\right| \leq CA\int_{\mathcal{R}} \left|\xi v^{\beta-\frac{1}{4}}\right|^2,$$

and together with the previous estimate, (4.6) and the fact that v is bounded below by (1.12), this shows that

$$\int_{\mathcal{R}} \left(\left|\zeta\partial_x v^\beta\right|^2 + k\left|\zeta\partial_y v^\beta\right|^2\right) \leq C\left(\frac{\beta}{2\beta-1}\right)^2 A^2\int_{\mathcal{R}} \left|\xi v^{\beta-\frac{1}{4}}\right|^2.$$

Using the one-dimensional Poincaré inequality again along with the inequality $\sqrt{v} \leq \sqrt{\frac{C}{k}} \leq \sqrt{\frac{C}{c}}$ on the support of ζ_x, we conclude that

$$\begin{aligned} \int_{\mathcal{R}} \left|\zeta v^\beta\right|^2 &\leq CR_1^2\int_{\mathcal{R}} \left|\partial_x\zeta v^\beta\right|^2 \leq CR_1^2\int_{\mathcal{R}} \left|\zeta\partial_x v^\beta\right|^2 + CR_1^2\int_{\mathcal{R}} \left|\zeta_x v^\beta\right|^2 \\ &\leq C\left(\frac{\beta}{2\beta-1}\right)^2 R_1^2A^2\int_{\mathcal{R}} \left|\xi v^{\beta-\frac{1}{4}}\right|^2. \end{aligned}$$

Applying this with successively $\beta = \frac{5}{4}, \frac{6}{4}, ..., \frac{12}{4}$, we obtain that

$$\int_{\mathcal{R}} \left|\zeta v^3\right|^2 \leq C \int_{\mathcal{R}} |\xi v|^2 \leq C \left\|\varkappa w\right\|_\infty^2 . \tag{4.8}$$

We now wish to show the third line in (4.3) is *under special control* by establishing Lemma 3.2 without assuming that $\|\zeta v\|_\infty$ is *under special control*, rather using only that $\|\xi v\|_{L^6}$ is *under special control* along with our hypothesis (1.11), and of course (1.12). The first step in proving Lemma 3.2 is the application of Corollary 2.7 with $k(x,y)$ replaced by $k(x, w(x,y))$, yielding an estimate for

$$\int_{\mathcal{R}} \left(|\partial_x(\zeta\partial\eta u)|^2 + k\,|\partial_y(\zeta\partial\eta u)|^2\right) + \int_{\mathcal{R}} \left(|\partial_x(\zeta\partial\eta v)|^2 + k\,|\partial_y(\zeta\partial\eta v)|^2\right). \tag{4.9}$$

Of course we really only need to estimate the integrals involving v. We remind the reader that k refers to $k(x, w(x,y))$ here. It is crucial to note that our hypothesis $|k_2| \leq Ck^{\frac{3}{2}}$ implies that $k(x, w(x,y))$ satisfies (1.5) - see (3.6). To illustrate the remaining argument, consider the revised estimate for the following term which arises in the proof just following (3.7):

$$\begin{aligned} \int_{\mathcal{R}} (\zeta\partial\eta\mathcal{L}v)(\zeta\partial\eta v) &= \int_{\mathcal{R}} \left(\zeta\partial\eta v^2\right)(k_2\partial_y\zeta\partial\eta v) - \int_{\mathcal{R}} \left(v^2\right)\left(\left[k_2\partial_y, \eta\partial\zeta^2\right]\partial\eta v\right) \\ &\quad - \int_{\mathcal{R}} \left(\zeta\partial\eta v^2\right)\left(k_2\zeta_y\partial\eta v\right) \\ &= I + II + III. \end{aligned} \tag{4.10}$$

For term I we use $|k_2| \leq Ck^{\frac{3}{2}}$ to get

$$|I| \leq \frac{C}{\varepsilon}\int_{\mathcal{R}} k^2 \left|\partial\eta v^2\right|^2 + \varepsilon\int_{\mathcal{R}} k\,|\partial_y\zeta\partial\eta v|^2 ,$$

and absorb the second term into (4.9) as usual. The first term now satisfies

$$\begin{aligned} \int_{\mathcal{R}} k^2 \left|\partial\eta v^2\right|^2 &\leq C\int_{\mathcal{R}} k^2\,|v\eta\partial v|^2 + C\int_{\mathcal{R}} k^2\left|(\partial\eta)\,v^2\right|^2 \\ &\leq C\,\|\eta v\|_1^2 + C\,\|(\partial\eta)\,v\|_{L^2}^2 \leq C\,\|\eta v\|_1^2 + CA^2\,\|\xi w\|_\infty , \end{aligned}$$

since $|kv| \leq C$ by (1.12) again, and since $\|\zeta v\|_2$ is *under special control*. The first term here is now absorbed into (4.9) by Poincaré's inequality as in Theorem 3.4. Term III is handled in similar fashion, using that $\sqrt{k}\partial\eta v$ already has L^2 norm *under special control*.

For term II we use

$$\left[k_2\partial_y, \eta\partial\zeta^2\right] = k_2\eta_y\partial\zeta^2 + k_2\eta\partial 2\zeta\zeta_y - \eta\zeta^2(\partial k_2)\,\partial_y,$$

to obtain

$$|II| \leq \left|\int_{\mathcal{R}} \left(v^2\right)\left(k_2\eta_y\partial\zeta^2\partial\eta v\right) + \int_{\mathcal{R}} \left(v^2\right)\left(k_2\eta\partial 2\zeta\zeta_y\partial\eta v\right)\right| + \left|\int_{\mathcal{R}} \left(v^2\right)\left(\eta\zeta^2(\partial k_2)\,\partial_y\partial\eta v\right)\right|.$$

Now using $|k_2| \leq Ck^{\frac{3}{2}}$, the first of the terms here satisfies

$$\begin{aligned}
&\left|\int_{\mathcal{R}} \left(v^2\right)\left(k_2 \eta_y \partial \zeta^2 \partial \eta v\right)\right| = \left|\int_{\mathcal{R}} \left(v^2\right)\left(k_2 \eta_y \zeta \partial \zeta \partial \eta v\right) + \int_{\mathcal{R}} \left(v^2\right)\left(k_2 \eta_y \left(\partial \zeta\right) \zeta \partial \eta v\right)\right| \\
&\quad \leq C\varepsilon \int_{\mathcal{R}} k \left|\partial \zeta \partial \eta v\right|^2 + \frac{C}{\varepsilon} \int_{\mathcal{R}} k^2 \left|\eta_y v^2\right|^2 + C \int_{\mathcal{R}} k^2 \left|\eta_y \left(\partial \zeta\right) v^2\right|^2 + C \int_{\mathcal{R}} \left|\partial \eta v\right|^2 \\
&\quad \leq C\varepsilon \int_{\mathcal{R}} k \left|\partial \zeta \partial \eta v\right|^2 + \frac{C}{\varepsilon} A^4 \left\|\xi v\right\|_{L^2}^2 + C \left\|\eta v\right\|_1^2,
\end{aligned}$$

where $0 < \varepsilon < 1$ since $kv \leq C$. The first term on the right above can be absorbed into (4.9), while the second term, $\|\xi v\|_{L^2}$, is *under special control*, and the third term, $\|\eta v\|_1^2$, can be absorbed into (4.9) by Poincaré's inequality in one variable. The term $\int_{\mathcal{R}} \left(v^2\right)\left(k_2 \eta \partial 2 \zeta \zeta_y \partial \eta v\right)$ is handled in the same way. Finally, the term involving ∂k_2 is handled separately for $\partial = \partial_x$ and $\partial = \partial_y$ as follows. In the case $\partial = \partial_x$, commuting one factor of ζ with ∂_x we obtain

$$\begin{aligned}
(4.11) \qquad &\left|\int_{\mathcal{R}} \left(v^2\right)\left(\eta \zeta^2 \left(\partial_x k_2\right) \partial_y \partial_x \eta v\right)\right| = \left|\int_{\mathcal{R}} \left(v^2\right)\left(\eta \zeta^2 \left(k_{21} + k_{22} u\right) \partial_y \partial_x \eta v\right)\right| \\
&\quad \leq \left|\int_{\mathcal{R}} \left(\eta \zeta v^2\right)\left(k_{21} \partial_x \zeta \partial_y \eta v\right)\right| + \left|\int_{\mathcal{R}} \left(\eta \zeta v^2\right)\left(k_{22} u \partial_x \zeta \partial_y \eta v\right)\right| \\
&\qquad + \left|\int_{\mathcal{R}} \left(v^2\right)\left(\eta \zeta \zeta_x \left(k_{21} + k_{22} u\right) \partial_y \eta v\right)\right|,
\end{aligned}$$

where the final term here is *under special control* since ζ_x is supported where k is bounded away from zero, and so by (1.12), where u and v are bounded. Indeed, the final term is at most

$$C \int_{\mathcal{R}} \left|\zeta_x v \left(\partial_y \eta v\right)\right| \leq \int_{\mathcal{R}} \frac{k}{c} \left|\xi v \left(\partial_y \eta v\right)\right| \leq C \int_{\mathcal{R}} k \left|\partial_y \eta v\right|^2 + C \int_{\mathcal{R}} k \left|\xi v\right|^2,$$

which is *under special control* by the first two lines of (4.3). The first two integrals on the right side of (4.11) are dominated by

$$\frac{C}{\alpha} \int_{\mathcal{R}} \left|\eta \zeta v^{\frac{5}{2}}\right|^2 + \alpha \int_{\mathcal{R}} \left|\partial_x \zeta \partial_y \eta v\right|^2,$$

since $|u| \leq C\sqrt{v}$ and $0 < c \leq v$. The second term here can be absorbed into (4.9) and the first is *under special control*.

Now we turn our attention to the case $\partial = \partial_y$. For the moment we will consider k to mean $k(x,y)$ and write k_2 for $\left(\partial_y k\right)(x,y)$, etc. We will need the fact that $|k_2| \leq Ck$ implies $|\nabla k_2| \leq Ck^{\frac{1}{2}}$. Indeed, $k - ck_2 \geq 0$ and so by (1.5),

$$\left|\nabla \left(k - ck_2\right)\right| \leq C\sqrt{k - ck_2} \leq C\sqrt{k}$$

which implies by (1.5) again,

$$c\left|\nabla k_2\right| \leq |\nabla k| + C\sqrt{k} \leq C\sqrt{k}.$$

This inequality holds for (x,y) in a compact subset of Ω, and so $\left|\nabla k_2\left(x, w(x,y)\right)\right| \leq Ck\left(x, w(x,y)\right)^{\frac{1}{2}}$ holds for (x,y) in a compact subset of the interior of $T\Omega$. With this we now have

$$\begin{aligned}
\left|\int_{\mathcal{R}} \left(v^2\right)\left(\eta \zeta^2 \left(\partial_y k_2\right) \partial_y \partial_y \eta v\right)\right| \quad &\leq \quad \left|\int_{\mathcal{R}} \left(v^2\right)\left(\eta \zeta^2 \left(k_{22} v\right) \partial_y \partial_y \eta v\right)\right| \\
&\leq \quad \left|\int_{\mathcal{R}} \left(\eta \zeta v^3\right)\left(\sqrt{k} \partial_y \zeta \partial_y \eta v\right)\right|,
\end{aligned}$$

plus a term $\int_{\mathcal{R}} |\eta\zeta v^3| \left|\sqrt{k}\zeta_y \partial_y \eta v\right|$ that is *under special control* by the Cauchy-Schwartz inequality, (4.8) and the second line of (4.3). We continue with

$$\left|\int_{\mathcal{R}} (\eta\zeta v^3)\left(\sqrt{k}\partial_y \zeta \partial_y \eta v\right)\right| \leq \frac{C}{\alpha}\int_{\mathcal{R}} \left|\eta\zeta v^3\right|^2 + \alpha \int_{\mathcal{R}} k \left|\partial_y \zeta \partial_y \eta v\right|^2$$

where the first term is *under special control* and the second can be absorbed into (4.9). Similar arguments handle the term $\int_{\mathcal{R}} (\zeta\partial\eta\mathcal{L}u)(\zeta\partial\eta u)$ in (3.7), except that this time we have $\zeta\partial\eta u = \zeta\partial_x \eta\partial w$, which is either $\zeta\partial_x\eta u$ or $\zeta\partial_x\eta v$, modulo terms whose L^2 norm is *under special control*. No factor of $\sqrt{k}$ is needed to absorb this term and so we only require the boundedness of the second order partial derivatives of k that arise here. The remaining terms in (3.7) are easily handled using (1.11), (1.12) and the terms already proven to be *under special control*, thereby establishing that (4.9) is *under special control*, and completing the proof of the third assertion in (4.3).

As indicated at the beginning, this completes the proof of Theorem 1.1.

4.2. **The generalized equation.**

Proof. (of Theorem 1.2) The proof is very similar to that of Theorem 1.1. The main points are that inequality (1.5) persists for $k(x,y,v,p,q)$ in a compact subset K of $\Omega\times\mathbb{R}^3$, and can be applied to the quasilinear equation (1.8) since $(x, w(x,y), r(x,y), z(x,y), y)$ lies in a compact subset of $\Omega\times\mathbb{R}^3$ for $(x,y)\in\mathcal{R}$, a compact subset of Ω', by the C^1 *a priori* estimates in say [1] (the proofs in this reference use only $k\geq 0$ for these estimates). Moreover, the gradients of the auxiliary functions r and z are expressible in terms of z and the gradient of w times smooth functions, namely from

$$r_x = z + yw_x, r_y = yw_y, z_x = kw_y, z_y = -w_x, \tag{4.12}$$

where k is evaluated at $(x, w(x,y), r(x,y), z(x,y), y)$. Thus ∇z satisfies the same estimates as does ∇w at any point in the argument, and then likewise for ∇r (recall that the sup norm bounds of both z and r appear on the right side of the conclusion of Theorem 1.2).

To illustrate, we consider the extension of Lemma 3.1 to the present setting. If we set

$$\mathcal{L} = \partial_x^2 + \partial_y \widetilde{k}(x,y)\partial_y$$

where $\widetilde{k}(x,y) = k(x, w(x,y), r(x,y), z(x,y), y)$, and differentiate the equation $\mathcal{L}(w) = 0$ with respect to y, we obtain

$$\mathcal{L}(\partial_y w) = -\partial_y\left(\partial_y \widetilde{k}\right)\partial_y w.$$

By using (4.12), we have

$$\partial_y \widetilde{k} = k_2 w_y + k_3 y w_y - k_4 w_x + k_5$$

where the partial derivatives k_j are evaluated at the point $(x, w(x,y), r(x,y), z(x,y), y)$. The key step in extending Lemma 3.1 is to estimate (with $v = \partial_y w$ and $u = \partial_x w$)

$$\begin{aligned} -\int_{\mathcal{R}} (\zeta \mathcal{L} v)(\zeta v) &= \int_{\mathcal{R}} \left(\zeta \partial_y \left(\partial_y \widetilde{k}\right) v\right)(\zeta v) = \int_{\mathcal{R}} (\zeta \partial_y (k_2 v + k_3 y v - k_4 u + k_5) v)(\zeta v) \\ &= \int_{\mathcal{R}} \left(\zeta \partial_y k_2 v^2\right)(\zeta v) + \int_{\mathcal{R}} \left(\zeta \partial_y k_3 y v^2\right)(\zeta v) \\ &\quad - \int_{\mathcal{R}} (\zeta \partial_y k_4 u v)(\zeta v) + \int_{\mathcal{R}} (\zeta \partial_y k_5 v)(\zeta v). \end{aligned}$$

The first term on the right is the only term appearing in the proof of Lemma 3.1, and it is evident that the same techniques apply to the remaining three terms. This completes our discussion of the proof of Theorem 1.2.

We now extend the argument in the previous section to prove Theorem 1.3. Additional considerations arise due to the interplay of partial derivatives of k and the derivatives of r and z in (4.12).

Proof. (of Theorem 1.3) We suppose that w is a smooth solution of (1.8). If $\widetilde{k}$ denotes the function $k(x, w(x,y), r(x,y), z(x,y), y)$, then a direct calcuation yields

$$\begin{aligned} \partial_x \widetilde{k} &= k_1 + k_2 w_x + k_3 (z + y w_x) + k_4 k w_y, \\ \partial_y \widetilde{k} &= k_2 w_y + k_3 y w_y - k_4 w_x + k_5. \end{aligned} \tag{4.13}$$

Just as in the section on quasilinear equations, we continue to write k in place of $\widetilde{k}$ and continue to use variables x and y, writing k_j with $j = 1,2,3,4,5$ to indicate partial derivatives of k with respect to the original 5 variables x, y, v, p, q as in (1.9). Thus for example, k_4 means (recall that $k = k(x,y,v,p,q)$)

$$(\partial_p k)(x, w(x,y), r(x,y), z(x,y), y).$$

We will say that an expression is *under special control* if it is dominated by

$$\mathcal{C}(L), \textit{ when } (x,w,r,z,y) \in L \textit{ compact } \subset \Omega \times \mathbb{R}^3.$$

We now establish analogues of the three successive assertions in (4.3). We have from (2.5), namely

$$\int_{\mathcal{R}} \left(|\zeta \partial_x w|^2 + \left|\zeta \sqrt{k} \partial_y w\right|^2\right) \leq -2 \int_{\mathcal{R}} (\zeta \mathcal{L} w)(\zeta w) + 4A^2 \int_{\mathcal{R}} |\varrho_1 w|^2 + 4A^2 \int_{\mathcal{R}} k |\varrho_2 w|^2,$$

and $\mathcal{L} w = 0$, that $\|\zeta \partial_x w\|_2 + \left\|\zeta \sqrt{k} \partial_y w\right\|_2 \leq C \|\xi w\|_\infty$, the analogue of the first line of (4.3). We now wish to estimate, writing $u = \partial_x w$ and $v = \partial_y w$ as usual,

$$\int_{\mathcal{R}} \left(|\zeta \partial_x u|^2 + k |\zeta \partial_y u|^2\right) + \int_{\mathcal{R}} \left(|\zeta \partial_x v|^2 + k |\zeta \partial_y v|^2\right). \tag{4.14}$$

Note that it is necessary to include the k-energy of u as well as v this time because the formulas in (4.13) each involve both u and v on the right hand side. Replacing w by v in (2.5) we obtain

$$\begin{aligned} \int_{\mathcal{R}} \left(|\zeta \partial_x v|^2 + k |\zeta \partial_y v|^2\right) &\leq -2 \int_{\mathcal{R}} (\zeta \mathcal{L} v)(\zeta v) + 4A^2 \int_{\mathcal{R}} |\varrho_1 v|^2 + 4A^2 \int_{\mathcal{R}} k |\varrho_2 v|^2 \\ &\leq -2 \int_{\mathcal{R}} (\zeta \mathcal{L} v)(\zeta v) + \mathbf{TUSC}, \end{aligned}$$

where **TUSC** stands for terms *under special control*. Indeed, the indicated integrals are **TUSC** since $\int_{\mathcal{R}} |\varrho_1 v|^2 \leq C \int_{\mathcal{R}} k |\varrho_1 v|^2$ (since $k \geq c > 0$ on the support of ϱ_1) and $\int_{\mathcal{R}} k |\varrho_j v|^2 \leq \left\| \xi \sqrt{k} v \right\|_{L^2}^2 \leq C \|\varkappa w\|_\infty^2$ by the earlier inequality. Next we compute

$$
\begin{aligned}
(4.15) \qquad \int_{\mathcal{R}} (\zeta \mathcal{L} v)(\zeta v) &= -\int_{\mathcal{R}} (\zeta \partial_y (\partial_y k) v)(\zeta v) \\
&= -\int_{\mathcal{R}} (\zeta \partial_y (k_2 v + k_3 y v - k_4 u + k_5) v)(\zeta v) \\
&= \int_{\mathcal{R}} v^2 k_2 (\partial_y \zeta^2 v) + \int_{\mathcal{R}} y v^2 k_3 (\partial_y \zeta^2 v) \\
&\quad - \int_{\mathcal{R}} u v k_4 (\partial_y \zeta^2 v) + \int_{\mathcal{R}} v k_5 (\partial_y \zeta^2 v).
\end{aligned}
$$

Now we use the hypotheses $\sqrt{k}|u| \leq C$ and $kv \leq C$ as well as $\partial_y \zeta^2 v = 2\zeta\zeta_y v + \zeta^2 \partial_y v$ and note that $2\sqrt{k}\zeta_y v$ has L^2 norm *under special control*, and that the L^2 norm of $\alpha\zeta\sqrt{k}\partial_y v$ can be absorbed into (4.14) for α sufficiently small. Moreover, we claim that $\frac{1}{\alpha}\int |\zeta v|^2$ is a sum of terms that can either be absorbed into (4.14) or are *under special control*. Indeed, by Poincaré's inequality in one variable,

$$
\frac{1}{\alpha}\int |\zeta v|^2 \leq C\frac{1}{\alpha}R_1^2 \int |\partial_x \zeta v|^2 \leq C\frac{1}{\alpha}R_1^2 \int |\zeta_x v|^2 + C\frac{1}{\alpha}R_1^2 \int |\zeta \partial_x v|^2 .
$$

The first term is *under special control* since $k \geq c > 0$ on the support of ζ_x, while the second can be absorbed for R_1 sufficiently small.

For the remainder of the proof of (4.14), we will say that a term can be *handled* if it can be decomposed into a sum of terms that can either be absorbed into (4.14) or are *under special control*. We will sometimes write **TUSCA** (**T**erms **U**nder **S**pecial **C**ontrol or **A**bsorbable) to designate terms that can be *handled*. From these observations, we now see that in (4.15), we need $|k_2| \leq Ck^{\frac{3}{2}}$ to *handle* $\int_{\mathcal{R}} \zeta v^2 k_2 (\zeta \partial_y v)$, $|k_3| \leq Ck^{\frac{3}{2}}$ to *handle* $\int_{\mathcal{R}} \zeta y v^2 k_3 (\zeta \partial_y v)$, $|k_4| \leq Ck$ to *handle* $\int_{\mathcal{R}} \zeta u v k_4 (\zeta \partial_y v)$, and $|k_5| \leq Ck^{\frac{1}{2}}$ to *handle* $\int_{\mathcal{R}} \zeta v k_5 (\zeta \partial_y v)$. The corresponding terms where ∂_y hits ζ are handled similarly. Now we turn to

$$
\begin{aligned}
\int_{\mathcal{R}} \left(|\zeta \partial_x u|^2 + k |\zeta \partial_y u|^2 \right) &\leq -2\int_{\mathcal{R}} (\zeta \mathcal{L} u)(\zeta u) + 4A^2 \int_{\mathcal{R}} |\varrho_1 u|^2 + 4A^2 \int_{\mathcal{R}} k |\varrho_2 u|^2 \\
&= -2\int_{\mathcal{R}} (\zeta \mathcal{L} u)(\zeta u) + \mathbf{TUSC},
\end{aligned}
$$

where the terms designated **TUSC** are *under special control* since $\|\zeta u\|_{L^2}$ is *under special control*. We thus compute

$$
\begin{aligned}
-\int_{\mathcal{R}} (\zeta \mathcal{L} u)(\zeta u) &= \int_{\mathcal{R}} (\zeta \partial_y (\partial_x k) v)(\zeta u) \\
&= \int_{\mathcal{R}} (\zeta \partial_y (k_1 + k_2 u + k_3 (z + yu) + k_4 k v) v)(\zeta u) \\
&= \int_{\mathcal{R}} v k_1 (\partial_y \zeta^2 u) + \int_{\mathcal{R}} u v k_2 (\partial_y \zeta^2 u) \\
&\quad + \int_{\mathcal{R}} (zv + yuv) k_3 (\partial_y \zeta^2 u) + \int_{\mathcal{R}} v^2 k_4 k (\partial_y \zeta^2 u).
\end{aligned}
$$

Now we use $\partial_y \zeta^2 u = 2\zeta \zeta_y u + \zeta^2 \partial_y u$ and note that $2\zeta_y u$ has L^2 norm *under special control*, and that $\alpha \zeta \partial_y u$ can be absorbed into (4.14) in L^2 norm for sufficiently small α if multiplied by a factor $\sqrt{k}$. Using $\sqrt{k}\,|u| \leq C$ and $kv \leq C$ together with the fact that $\int |\zeta v|^2$ can be *handled,* we see that all of the above integrals are **TUSCA**. This completes the proof that (4.14) is *under special control*, the analogue of the second line of (4.3).

Altogether we now have that

$$\|\xi \partial_x v\|_{L^2} + \left\| \xi \sqrt{k} \partial_y v \right\|_{L^2} + \|\xi \partial_x u\|_{L^2} = \mathbf{TUSC}. \tag{4.16}$$

By the Poincaré inequality in one variable, together with $k \geq c > 0$ on the support of ζ_x, and $|u| \leq C\sqrt{v}$, we also have that

$$\|\xi v\|_{L^2} + \|\xi u\|_{L^4} = \mathbf{TUSC}. \tag{4.17}$$

The next step, following the proof in the previous section, is to use the Moser interation technique to show that $\|\zeta v\|_{L^6}$ is *under special control*. The inequality in Lemma 2.8 yields

$$\begin{aligned} \int_{\mathcal{R}} \left(\left|\zeta \partial_x v^\beta\right|^2 + k \left|\zeta \partial_y v^\beta\right|^2 \right) &\leq \frac{2\beta^2}{2\beta - 1} \left| \int_{\mathcal{R}} (\zeta \mathcal{L} v) \left(\zeta v^{2\beta - 1}\right) \right| \\ &+ \left(\frac{2\beta}{2\beta-1}\right)^2 A^2 \int_{\mathcal{R}} \left|\varrho_1 v^\beta\right|^2 + \left(\frac{2\beta}{2\beta-1}\right)^2 A^2 \int_{\mathcal{R}} k \left|\varrho_2 v^\beta\right|^2 , \end{aligned} \tag{4.18}$$

for $\beta > \frac{1}{2}$. Now $k \geq c > 0$ on the support of ϱ_1, and so $\int_{\mathcal{R}} \left|\varrho_1 v^\beta\right|^2 \leq C \int_{\mathcal{R}} k \left|\varrho_1 v^\beta\right|^2$. As a result of this together with (1.12), the second and third terms on the right above are dominated by

$$C \left(\frac{\beta}{2\beta - 1}\right)^2 A^2 \int_{\mathcal{R}} \left|\xi v^{\beta - \frac{1}{2}}\right|^2 .$$

We will now show that the first term is bounded by a similar integral plus terms which can be absorbed. We have

$$\begin{aligned} \int_{\mathcal{R}} (\zeta \mathcal{L} v) \left(\zeta v^{2\beta-1}\right) &= - \int_{\mathcal{R}} \left(\zeta \partial_y \left(\partial_y k\right) v\right) \left(\zeta v^{2\beta-1}\right) \\ &= - \int_{\mathcal{R}} \left(\zeta \partial_y \left(k_2 v + k_3 y v - k_4 u + k_5\right) v\right) \left(\zeta v^{2\beta-1}\right) \\ &= \int_{\mathcal{R}} v^2 k_2 \left(\partial_y \zeta^2 v^{2\beta-1}\right) + \int_{\mathcal{R}} y v^2 k_3 \left(\partial_y \zeta^2 v^{2\beta-1}\right) \\ &\quad - \int_{\mathcal{R}} u v k_4 \left(\partial_y \zeta^2 v^{2\beta-1}\right) + \int_{\mathcal{R}} v k_5 \left(\partial_y \zeta^2 v^{2\beta-1}\right) . \end{aligned} \tag{4.19}$$

The first integral on the right satisfies

$$\left| \int_{\mathcal{R}} v^2 k_2 \left(\partial_y \zeta^2 v^{2\beta-1}\right) \right| = \left| \int_{\mathcal{R}} \left(\zeta k_2 v^2\right) \left(\zeta \partial_y v^{2\beta-1}\right) + 2 \int_{\mathcal{R}} \left(\zeta_y k_2 v^2\right) \left(\zeta v^{2\beta-1}\right) \right| ,$$

and just after (4.7) in subsection 4.1, we showed that the first integral here satisfies

$$\begin{aligned} \left| \int_{\mathcal{R}} \left(\zeta k_2 v^2\right) \left(\zeta \partial_y v^{2\beta-1}\right) \right| &\leq \frac{C}{\alpha} \int_{\mathcal{R}} \left|\zeta v^\beta\right|^2 + C\alpha \int_{\mathcal{R}} k \left|\zeta \partial_y v^\beta\right|^2 \\ &\leq \frac{C}{\alpha} R_1^2 \int_{\mathcal{R}} \left|\zeta \partial_x v^\beta\right|^2 + \frac{C}{\alpha} A^2 R_1^2 \int_{\mathcal{R}} \left|\xi v^{\beta - \frac{1}{2}}\right|^2 + C\alpha \int_{\mathcal{R}} k \left|\zeta \partial_y v^\beta\right|^2 , \end{aligned}$$

since $0 < c \leq k \leq Cv^{-1}$ on the support of ζ_x, while the second integral is at most

$$C \int_{\mathcal{R}} k^{\frac{3}{2}} \left| \zeta_y \zeta v^{2\beta+1} \right| \leq CA \int_{\mathcal{R}} \left| \xi v^{\beta - \frac{1}{4}} \right|^2 .$$

Using the inequalities $|k_3| \leq Ck^{\frac{3}{2}}$, $|k_4| \leq Ck$ and $|k_5| \leq Ck^{\frac{1}{2}}$, we can show similar estimates for the remaining terms in (4.19), and then absorbing the relevant terms into the left side of (4.18) yields

$$\int_{\mathcal{R}} \left(\left| \zeta \partial_x v^\beta \right|^2 + k \left| \zeta \partial_y v^\beta \right|^2 \right) \leq \mathcal{C}(\beta) A^2 \int_{\mathcal{R}} \left| \xi v^{\beta - \frac{1}{4}} \right|^2 .$$

Using the one-dimensional Poincaré inequality again, we conclude that

$$\int_{\mathcal{R}} \left| \zeta v^\beta \right|^2 \leq CR_1^2 \int_{\mathcal{R}} \left| \partial_x \zeta v^\beta \right|^2 \leq CR_1^2 \int_{\mathcal{R}} \left| \zeta \partial_x v^\beta \right|^2 + CR_1^2 \int_{\mathcal{R}} \left| \zeta_x v^\beta \right|^2 \leq \mathcal{C}(\beta) R_1^2 A^2 \int_{\mathcal{R}} \left| \xi v^{\beta - \frac{1}{4}} \right|^2 .$$

Applying this with successively $\beta = \frac{5}{4}, \frac{6}{4}, ..., \frac{12}{4}$, we obtain that $\int_{\mathcal{R}} \left| \zeta v^3 \right|^2 \leq C \left\| \varkappa w \right\|_\infty^2$.

We now wish to show the analogue of the third line in (4.3). As in (4.9) we estimate

$$\int_{\mathcal{R}} \left(\left| \partial_x (\zeta \nabla \eta u) \right|^2 + k \left| \partial_y (\zeta \nabla \eta u) \right|^2 \right) + \int_{\mathcal{R}} \left(\left| \partial_x (\zeta \nabla \eta v) \right|^2 + k \left| \partial_y (\zeta \nabla \eta v) \right|^2 \right) . \tag{4.20}$$

In order to apply Corollary 2.7, we need $|\partial_x k| + |\partial_y k| \leq Ck^{\frac{1}{2}}$. From (4.13) and $|u|^2 \leq Cv \leq Ck^{-1}$, we see that this in fact holds provided $|k_i| \leq Ck^{d(i)}$ with $d(i) = \frac{3}{2}$ for $i = 2$ and 3, 1 for $i = 4$, $\frac{1}{2}$ for $i = 5$ and 1. This allows us to complete the estimation of all terms which result from Corollary 2.7, except the main terms involving $\mathcal{L}u$ and $\mathcal{L}v$. To estimate these main terms, we begin by using

$$\begin{aligned} \mathcal{L}u &= -\partial_y \left\{ k_1 v + k_2 uv + k_3 (zv + yuv) + k_4 kv^2 \right\}, \\ \mathcal{L}v &= -\partial_y \left\{ k_2 v^2 + k_3 yv^2 - k_4 uv + k_5 v \right\}, \end{aligned}$$

to obtain

$$\begin{aligned} \int_{\mathcal{R}} (\zeta \partial \eta \mathcal{L}u)(\zeta \partial \eta u) &= -\int_{\mathcal{R}} \left(\zeta \partial \eta \partial_y \left\{ k_1 v + k_2 uv + k_3 (zv + yuv) + k_4 kv^2 \right\} \right) (\zeta \partial \eta u) \\ &= I^u + II^u + III^u, \end{aligned}$$

and

$$\begin{aligned} \int_{\mathcal{R}} (\zeta \partial \eta \mathcal{L}v)(\zeta \partial \eta v) &= -\int_{\mathcal{R}} \left(\zeta \partial \eta \partial_y \left\{ k_2 v^2 + k_3 yv^2 - k_4 uv + k_5 v \right\} \right) (\zeta \partial \eta v) \\ &= I^v + II^v + III^v, \end{aligned}$$

where the decompositions into $I^u + II^u + III^u$ and $I^v + II^v + III^v$ are as in (4.10), now forming a commutator for each $\partial_y k_i$. We have

$$\begin{aligned} I^u &= \int_{\mathcal{R}} (\zeta \partial \eta v) k_1 (\partial_y \zeta \partial \eta u) + \int_{\mathcal{R}} (\zeta \partial \eta uv) k_2 (\partial_y \zeta \partial \eta u) \\ &\quad + \int_{\mathcal{R}} (\zeta \partial \eta (zv + yuv)) k_3 (\partial_y \zeta \partial \eta u) + \int_{\mathcal{R}} \left(\zeta \partial \eta v^2 \right) k_4 k (\partial_y \zeta \partial \eta u) \end{aligned}$$

and

$$\begin{aligned} I^v &= \int_{\mathcal{R}} \left(\zeta \partial \eta v^2 \right) k_2 (\partial_y \zeta \partial \eta v) + \int_{\mathcal{R}} \left(\zeta \partial \eta y v^2 \right) k_3 (\partial_y \zeta \partial \eta v) \\ &\quad - \int_{\mathcal{R}} (\zeta \partial \eta uv) k_4 (\partial_y \zeta \partial \eta v) + \int_{\mathcal{R}} (\zeta \partial \eta v) k_5 (\partial_y \zeta \partial \eta v) . \end{aligned}$$

To handle these terms, we note that $\alpha\sqrt{k}\partial_y\zeta\partial\eta u$ and $\alpha\sqrt{k}\partial_y\zeta\partial\eta v$ have L^2 norms that can be absorbed into (4.20) for sufficiently small α. We can also absorb the L^2 norms of terms of the form $\frac{1}{\alpha}\zeta\partial\eta u$ and $\frac{1}{\alpha}\zeta\partial\eta v$ by using the Poincaré inequality in one variable. Finally, we see that all of the above terms can be handled with our hypotheses on k_j by manipulations of the form

$$\zeta\partial\eta v^2 = 2v\left(\zeta\partial\eta v\right) - \zeta\left(\partial\eta\right)v^2,$$

and

$$\partial_y\zeta\partial\eta u = \left[\partial_y, \zeta\partial\eta\right]u - \left[\partial_x, \zeta\partial\eta\right]v + \partial_x\zeta\partial\eta v,$$

where we have used $\partial_y u = \partial_x v$. For example, the first equality renders the first integral on the right side of I^v tractable as follows:

$$\begin{aligned}
\left|\int_{\mathcal{R}}\left(\zeta\partial\eta v^2\right)k_2\left(\partial_y\zeta\partial\eta v\right)\right| &\leq \left|\int_{\mathcal{R}}\left(2v\left(\zeta\partial\eta v\right)\right)k_2\left(\partial_y\zeta\partial\eta v\right)\right| + \left|\int_{\mathcal{R}}\left(\zeta\left(\partial\eta\right)v^2\right)k_2\left(\partial_y\zeta\partial\eta v\right)\right| \\
&\leq \int_{\mathcal{R}}\left(2v\left|\zeta\partial\eta v\right|\right)k^{\frac{3}{2}}\left|\partial_y\zeta\partial\eta v\right| + \int_{\mathcal{R}}\left|\zeta\left(\partial\eta\right)v^2\right|k^{\frac{3}{2}}\left|\partial_y\zeta\partial\eta v\right| \\
&\leq \int_{\mathcal{R}}\left(2\left|\zeta\partial\eta v\right|\right)k^{\frac{1}{2}}\left|\partial_y\zeta\partial\eta v\right| + \int_{\mathcal{R}}\left|\zeta\left(\partial\eta\right)v\right|k^{\frac{1}{2}}\left|\partial_y\zeta\partial\eta v\right|.
\end{aligned}$$

Here we can absorb $\left\|k^{\frac{1}{2}}\partial_y\zeta\partial\eta v\right\|_{L^2}$ and then use Poincaré in one variable to absorb $\left\|\zeta\partial\eta v\right\|_{L^2}$, and finally note that $\left\|\zeta\left(\partial\eta\right)v\right\|_{L^2}$ is *under special control*. The second identity renders the third integral on the right side of I^v tractable as follows:

$$\begin{aligned}
\int_{\mathcal{R}}\left(\zeta\partial\eta uv\right)k_4\left(\partial_y\zeta\partial\eta v\right) &= \int_{\mathcal{R}}\left(\zeta\partial\eta uv\right)k_4\left(\left[\partial_y, \zeta\partial\eta\right]u\right) \\
&\quad - \int_{\mathcal{R}}\left(\zeta\partial\eta uv\right)k_4\left(\left[\partial_x, \zeta\partial\eta\right]v\right) + \int_{\mathcal{R}}\left(\zeta\partial\eta uv\right)k_4\left(\partial_x\zeta\partial\eta v\right).
\end{aligned}$$

Each of the terms

$$\left[\partial_y, \zeta\partial\eta\right]u = \zeta_y\partial\eta u + \zeta\partial\eta_y u \text{ and } \left[\partial_x, \zeta\partial\eta\right]v = \zeta_x\partial\eta v + \zeta\partial\eta_x v$$

lies in L^2 *under special control* since $\partial u = \partial_x\partial w \in L^2$ *under special control* by (4.16) and (4.17), and ζ_x and η_x are supported where $k \geq c > 0$. Moreover $\partial_x\zeta\partial\eta v$ has L^2 norm that can be absorbed into (4.20). Then we can use

$$\begin{aligned}
\int_{\mathcal{R}}\left|\left(\zeta\partial\eta uv\right)k_4\right|^2 &\leq \int_{\mathcal{R}}\left|\left(\zeta\partial\eta u\right)vk\right|^2 + \int_{\mathcal{R}}\left|\left(\zeta\partial\eta v\right)uk\right|^2 + \mathbf{TUSC} \\
&\leq C\int_{\mathcal{R}}\left|\left(\zeta\partial\eta u\right)\right|^2 + \int_{\mathcal{R}}k\left|\left(\zeta\partial\eta v\right)\right|^2 + \mathbf{TUSC} \\
&\leq C\int_{\mathcal{R}}\left|\left(\zeta\partial_x\eta\partial w\right)\right|^2 + \int_{\mathcal{R}}k\left|\left(\zeta\partial\eta v\right)\right|^2 + \mathbf{TUSC}.
\end{aligned}$$

Thus by (4.16) and (4.17) all of the terms in I^u and I^v are now *under special control*.

The type III terms are given by

$$\begin{aligned}
III^u &= \int_{\mathcal{R}}\left(\zeta\partial\eta v\right)k_1\left(\zeta_y\partial\eta u\right) + \int_{\mathcal{R}}\left(\zeta\partial\eta uv\right)k_2\left(\zeta_y\partial\eta u\right) \\
&\quad + \int_{\mathcal{R}}\left(\zeta\partial\eta\left(zv + yuv\right)\right)k_3\left(\zeta_y\partial\eta u\right) + \int_{\mathcal{R}}\left(\zeta\partial\eta v^2\right)k_4 k\left(\zeta_y\partial\eta u\right),
\end{aligned}$$

and

$$\begin{aligned} III^v &= \int_{\mathcal{R}} \left(\zeta\partial\eta v^2\right) k_2 \left(\zeta_y \partial\eta u\right) + \int_{\mathcal{R}} \left(\zeta\partial\eta y v^2\right) k_3 \left(\zeta_y \partial\eta u\right) \\ &\quad - \int_{\mathcal{R}} \left(\zeta\partial\eta u v\right) k_4 \left(\zeta_y \partial\eta u\right) + \int_{\mathcal{R}} \left(\zeta\partial\eta v\right) k_5 \left(\zeta_y \partial\eta u\right), \end{aligned}$$

and are handled in similar fashion to the type I terms.

We now turn to the type II terms, which require the hypotheses $|k_{ij}| \leq Ck^{\frac{1}{2}}$ for $2 \leq i,j \leq 5$. Since by the hypothesis (1.10) we already have $|k_j| \leq Ck$ for $2 \leq j \leq 4$, it follows from (1.6) as before that if $ck_j \leq k$, then

$$c\left|\nabla k_j\right| = \left|\nabla k - \nabla\left(k - ck_j\right)\right| \leq |\nabla k| + \left|\nabla\left(k - ck_j\right)\right| \leq C\sqrt{k} + C\sqrt{\left(k - ck_j\right)} \leq Ck^{\frac{1}{2}}$$

for $2 \leq j \leq 4$. Thus $|k_{55}| \leq Ck^{\frac{1}{2}}$ is the only second derivative estimate that must be assumed in the hypotheses. We begin with the identities

$$\begin{aligned} \left[k_j\partial_y, \eta\partial\zeta^2\right] &= k_j\eta_y\partial\zeta^2 + k_j\eta\partial 2\zeta\zeta_y - \eta\zeta^2\left(\partial k_j\right)\partial_y \\ &= A_j + B_j - C_j, \end{aligned}$$

for $1 \leq i,j \leq 5$. We will write $II^u = II^u_A + II^u_B + II^u_C$, $II^u_A = \sum_{j=1}^4 II^u_{A_j}$ and $II^v_A = \sum_{j=2}^5 II^v_{A_j}$ etc. with the obvious meanings. We have

$$\begin{aligned} II^u_A &= \int_{\mathcal{R}} vA_1\partial\eta u + \int_{\mathcal{R}} uvA_2\partial\eta u + \int_{\mathcal{R}} (zv + yuv)\,A_3\partial\eta u + \int_{\mathcal{R}} kv^2 A_4\partial\eta u \\ &= \int_{\mathcal{R}} vk_1\eta_y\partial\zeta^2\partial\eta u + \int_{\mathcal{R}} uvk_2\eta_y\partial\zeta^2\partial\eta u \\ &\quad + \int_{\mathcal{R}} (zv + yuv)\,k_3\eta_y\partial\zeta^2\partial\eta u + \int_{\mathcal{R}} kv^2 k_4\eta_y\partial\zeta^2\partial\eta u \end{aligned}$$

and these terms can be decomposed into terms that are either *under special control*, or can be absorbed into (4.20) after applying the Poincaré inequality in one variable. Indeed, since $\partial u = \partial_x\partial w$, we have $\partial\zeta\partial\eta u = \partial_x\zeta\partial\eta\left(\partial w\right)$ modulo terms of the form $\partial_x u$, $\partial_y u$, $\partial_x v$ and $\partial_y v$ with appropriate cutoff functions included, and terms involving only u, v or w with cutoff functions. Now the terms of the form $\partial_x u$ and $\partial_y u = \partial_x v$ have L^2 norm *under special control* since the integrals in (4.14) are *under special control*. The remaining term $\partial_y v$ can be absorbed into (4.20) after applying the Poincaré inequality in one variable so as to obtain the L^2 norm of $\partial_x\partial_y v$ (with cutoff functions) multiplied by R_1^2. Finally the main term $\partial_x\zeta\partial\eta\left(\partial w\right)$ has L^2 norm that can be absorbed into (4.20) if multiplied by a sufficiently small α. Similarly the term II^u_B is *under special control*.

Turning to the term II^u_C we have

$$\begin{aligned} II^u_C &= \int_{\mathcal{R}} vC_1\partial\eta u + \int_{\mathcal{R}} uvC_2\partial\eta u + \int_{\mathcal{R}} (zv + yuv)\,C_3\partial\eta u + \int_{\mathcal{R}} kv^2 C_4\partial\eta u \\ &= \int_{\mathcal{R}} v\eta\zeta^2\left(\partial k_1\right)\partial_y\partial\eta u + \int_{\mathcal{R}} uv\eta\zeta^2\left(\partial k_2\right)\partial_y\partial\eta u \\ &\quad + \int_{\mathcal{R}} (zv + yuv)\,\eta\zeta^2\left(\partial k_3\right)\partial_y\partial\eta u + \int_{\mathcal{R}} kv^2\eta\zeta^2\left(\partial k_4\right)\partial_y\partial\eta u. \end{aligned}$$

We now need the second derivatives,

$$\begin{aligned}\partial_x k_j &= \partial_x k_j (x, w, r, z, y) = k_{j1} + k_{j2} u + k_{j3} (z + yu) + k_{j4} k v, \\ \partial_y k_j &= \partial_y k_j (x, w, r, z, y) = k_{j2} v + k_{j3} y v - k_{j4} u + k_{j5}.\end{aligned}$$

Considering first the case $\partial = \partial_x$, we compute

$$\begin{aligned}II^u_{C_1} &= \int_{\mathcal{R}} v C_1 \partial \eta u = \int_{\mathcal{R}} v \eta \zeta^2 (\partial_x k_1) \partial_y \partial \eta u \\ &= \int_{\mathcal{R}} v \eta \zeta^2 (k_{11}) \partial_y \partial \eta u + \int_{\mathcal{R}} v \eta \zeta^2 (k_{12} u) \partial_y \partial \eta u \\ &\quad + \int_{\mathcal{R}} v \eta \zeta^2 (k_{13} (z + yu)) \partial_y \partial \eta u + \int_{\mathcal{R}} v \eta \zeta^2 (k_{14} k v) \partial_y \partial \eta u.\end{aligned}$$

Now since $\partial_y u = \partial_x v$, we can write as above $\zeta \partial_y \partial \eta u = \partial_x \zeta \partial \eta v$ modulo terms either with L^2 norm *under special control*, or that can be absorbed into (4.20) after applying the Poincaré inequality in one variable. Since the term $\partial_x \zeta \partial \eta v$ has L^2 norm that can be absorbed into (4.20) if multiplied by a sufficiently small α, and $\|\xi v\|_{L^6}$ is *under special control*, we see that we need $|k_{ij}| \leq C$ to obtain that $II^u_{C_1}$ is *under special control*. Similarly, we have $II^u_{C_\ell}$ is *under special control* for all ℓ provided $|k_{ij}| \leq C$. This completes the proof that II^u_C is *under special control* in the case $\partial = \partial_x$.

Turning now to the case $\partial = \partial_y$, we compute

$$\begin{aligned}II^u_{C_1} &= \int_{\mathcal{R}} v C_1 \partial \eta u = \int_{\mathcal{R}} v \eta \zeta^2 (\partial_y k_1) \partial_y \partial \eta u \\ &= \int_{\mathcal{R}} v \eta \zeta^2 (k_{12} v) \partial_y \partial \eta u + \int_{\mathcal{R}} v \eta \zeta^2 (k_{13} y v) \partial_y \partial \eta u \\ &\quad - \int_{\mathcal{R}} v \eta \zeta^2 (k_{14} u) \partial_y \partial \eta u + \int_{\mathcal{R}} v \eta \zeta^2 (k_{15}) \partial_y \partial \eta u.\end{aligned}$$

Again we can write $\zeta \partial_y \partial_y \eta u = \partial_x \zeta \partial_y \eta v$ modulo terms either with L^2 norm *under special control*, or that can be absorbed into (4.20) after applying the Poincaré inequality in one variable. Since the term $\partial_x \zeta \partial_y \eta v$ has L^2 norm that can be absorbed into (4.20) if multiplied by a sufficiently small α, we see that $II^u_{C_1}$ is *under special control* since $\|\xi v\|_{L^6}$ is. Similarly, we have $II^u_{C_\ell}$ is *under special control* for all ℓ provided $|k_{ij}| \leq C$. This completes the proof that II^u_C is *under special control* in the case $\partial = \partial_y$.

We now investigate the corresponding estimates for II^v_A, II^v_B and II^v_C. We have

$$\begin{aligned}II^v_A &= \int_{\mathcal{R}} v^2 A_2 \partial \eta v + \int_{\mathcal{R}} y v^2 A_3 \partial \eta v - \int_{\mathcal{R}} u v A_4 \partial \eta v + \int_{\mathcal{R}} v A_5 \partial \eta v \\ &= \int_{\mathcal{R}} v^2 k_2 \eta_y \partial \zeta^2 \partial \eta v + \int_{\mathcal{R}} y v^2 k_3 \eta_y \partial \zeta^2 \partial \eta v \\ &\quad - \int_{\mathcal{R}} u v k_4 \eta_y \partial \zeta^2 \partial \eta v + \int_{\mathcal{R}} v k_5 \eta_y \partial \zeta^2 \partial \eta v.\end{aligned}$$

Now $\sqrt{k} \partial \zeta \partial \eta v$ can be absorbed into (4.20) if multiplied by a sufficiently small α, $kv \leq C$, ξv lies in L^2 *under special control*, and the L^2 norm of $\zeta \partial \eta v$ can be absorbed using Poincaré's inequality in one variable. Thus we see that we need $|k_2| \leq C k^{\frac{3}{2}}$, $|k_3| \leq C k^{\frac{3}{2}}$, $|k_4| \leq C k$ and $|k_5| \leq C k^{\frac{1}{2}}$ in order to have II^v_A *under special control*. Similarly the term II^v_B is *under special control*.

Turning to the term II_C^v we have

$$\begin{aligned} II_C^v &= \int_{\mathcal{R}} v^2 C_2 \partial \eta v + \int_{\mathcal{R}} y v^2 C_3 \partial \eta v - \int_{\mathcal{R}} u v C_4 \partial \eta v + \int_{\mathcal{R}} v C_5 \partial \eta v \\ &= \int_{\mathcal{R}} v^2 \eta \zeta^2 \left(\partial k_2\right) \partial_y \partial \eta v + \int_{\mathcal{R}} y v^2 \eta \zeta^2 \left(\partial k_3\right) \partial_y \partial \eta v \\ &\quad - \int_{\mathcal{R}} u v \eta \zeta^2 \left(\partial k_4\right) \partial_y \partial \eta v + \int_{\mathcal{R}} v \eta \zeta^2 \left(\partial k_5\right) \partial_y \partial \eta v. \end{aligned}$$

We recall the second derivatives,

$$\begin{aligned} \partial_x k_j &= \partial_x k_j \left(x, w, r, z, y\right) = k_{j1} + k_{j2} u + k_{j3} \left(z + y u\right) + k_{j4} k v, \\ \partial_y k_j &= \partial_y k_j \left(x, w, r, z, y\right) = k_{j2} v + k_{j3} y v - k_{j4} u + k_{j5}. \end{aligned}$$

Considering first the case $\partial = \partial_x$, we compute

$$\begin{aligned} II_{C_2}^v &= \int_{\mathcal{R}} v^2 C_2 \partial \eta v = \int_{\mathcal{R}} v^2 \eta \zeta^2 \left(\partial_x k_2\right) \partial_y \partial \eta v \\ &= \int_{\mathcal{R}} v^2 \eta \zeta^2 \left(k_{21}\right) \partial_y \partial \eta v + \int_{\mathcal{R}} v^2 \eta \zeta^2 \left(k_{22} u\right) \partial_y \partial \eta v \\ &\quad + \int_{\mathcal{R}} v^2 \eta \zeta^2 \left(k_{23} \left(z + y u\right)\right) \partial_y \partial \eta v + \int_{\mathcal{R}} v^2 \eta \zeta^2 \left(k_{24} k v\right) \partial_y \partial \eta v. \end{aligned}$$

Now $\zeta \partial_y \partial_x \eta v = \partial_x \zeta \partial_y \eta v$ modulo a term that is *under special control* by (4.16) since it is supported where $k \geq c > 0$. Since $\partial_x \zeta \partial_y \eta v$ has L^2 norm that can be absorbed into (4.20) if multiplied by a sufficiently small α, and since $\|\xi v\|_{L^6}$ is *under special control*, we see that $II_{C_2}^v$ is *under special control*. Similarly, we have $II_{C_\ell}^u$ is *under special control* for all ℓ provided $|k_{ij}| \leq C$. This completes the proof that II_C^v is *under special control* in the case that $\partial = \partial_x$.

Turning now to the final case $\partial = \partial_y$, we compute

$$\begin{aligned} II_{C_2}^v &= \int_{\mathcal{R}} v^2 C_2 \partial \eta v = \int_{\mathcal{R}} v^2 \eta \zeta^2 \left(\partial_y k_2\right) \partial_y \partial \eta v \\ &= \int_{\mathcal{R}} v^2 \eta \zeta^2 \left(k_{22} v\right) \partial_y \partial \eta v + \int_{\mathcal{R}} v^2 \eta \zeta^2 \left(k_{23} y v\right) \partial_y \partial \eta v \\ &\quad - \int_{\mathcal{R}} v^2 \eta \zeta^2 \left(k_{24} u\right) \partial_y \partial \eta v + \int_{\mathcal{R}} v^2 \eta \zeta^2 \left(k_{25}\right) \partial_y \partial \eta v. \end{aligned}$$

This time we need an additional factor of $\sqrt{k}$ to go with $\partial_y \zeta \partial_y \eta v$ so that $\sqrt{k} \partial_y \zeta \partial_y \eta v$ has L^2 norm that can be absorbed into (4.20) if multiplied by a sufficiently small α. Since $\|\xi v\|_{L^6}$ is *under special control*, we see that we only need $|k_{2j}| \leq C k^{\frac{1}{2}}$ for $2 \leq j \leq 5$ in order to have $II_{C_2}^v$ *under special control*. As mentioned above, these follow from our assumption that $|k_2| \leq C k^{\frac{3}{2}}$. Similarly, we have $II_{C_i}^v$ *under special control* for $i = 3, 4, 5$ if $|k_{ij}| \leq C k^{\frac{1}{2}}$ for $2 \leq i, j \leq 5$. Again, this follows from our assumptions on k_2, k_3, and k_4, with the exception of $|k_{55}| \leq C k^{\frac{1}{2}}$, which is part of the hypotheses. This completes the proof that II_C^v is *under special control* in the case $\partial = \partial_y$, and with this, the proof that the main terms in the application of Corollary 2.7 to (4.20) are *under special control*. The remaining terms are easier to handle, and then just as in the previous section, we conclude that $\|\zeta u\|_{L^\infty}$ and $\|\zeta v\|_{L^\infty}$ are *under special control*. Theorem 1.2 now completes the proof of Theorem 1.3.

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McMaster University, Hamilton, Ontario
E-mail address: sawyer@mcmaster.ca

Rutgers University, New Brunswick, N.J.
E-mail address: wheeden@math.rutgers.edu